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ANALYTICAL GEOMETRY

FOR BEGINNERS.

PART I.

THE STRAIGHT LINE AND CIRCLE.

BY THE

REV. T. G. VYVYAN, M.A.

MATHEMATICAL MASTER OF CHARTERHOUSE,
FORMERLY FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

LONDON:
GEORGE BELL AND SONS.
CAMBRIDGE: DEIGHTON, BELL AND CO.
NEW YORK: 66, FIFTH AVENUE.
1894

PREFACE.

THIS little book is an attempt to simplify a subject which the majority of boys find unusually difficult.

It is intended partly for young boys with some mathematical taste, and partly for those who have to get up a little for military and other examinations, but who do not intend to pursue their studies farther than is necessary for these examinations.

With these objects, the explanations have been made very full, but all reference to oblique coordinates and to transformations has been postponed to the last chapter.

A large number of easy examples have been specially prepared for this book.

Though some of the articles and examples have been taken from my Elementary Analytical Geometry, this book is constructed on a different plan from the other.

I hope that it will be found sufficient as far as it goes, and an easy introduction to more advanced treatises on the subject.

T. G. VYVYAN.

CHARTERHOUSE,
January, 1894.

CONTENTS.

CHAPTER VIII.

CHAPTER IX.

ANALYTICAL GEOMETRY.

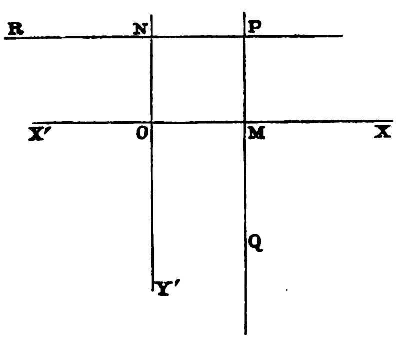
ERRATA.

Page 39, line 16, *for* $\frac{4p+3q}{7}$, $\frac{3p+4q}{7}$ *read* $\frac{4p+3q}{p+q}$, $\frac{3p+4q}{p+q}$.

" 46, line 20, *for* $\frac{3m-9}{4-9m}$ *read* $\frac{m-3}{1-3m}$.

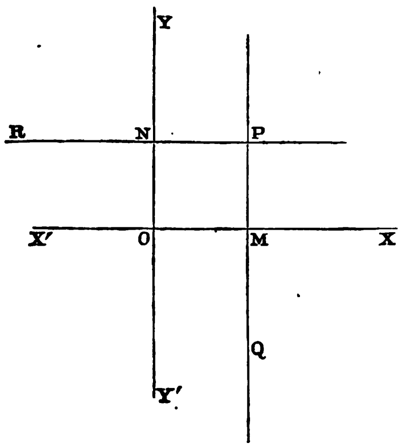
" 68, question 18, *for* 5 *read* 15.

" 84, question 17, *for* $+2by$ *read* $-2by$.



in the plane: through O draw any straight line XOX' of unlimited length, and draw YOY', also of unlimited length, perpendicular to XOX': in OX take any point M, and through M draw a straight line parallel to YOY'; it is evident that all points in this line are at the same distance OM from YOY'.

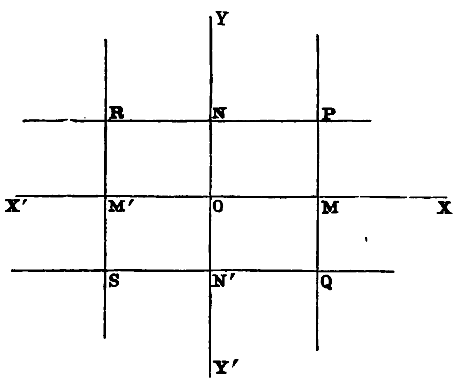
Let us denote distance from YOY' by the symbol x; then all points in the line PMQ have the same value of x, and if we call that value a, the equation $x = a$ is true for all points on that line.



Similarly, if through any point N on YOY' we draw RNP parallel to XOX', all points in this line are at the same distance from XOX'; and if we denote distance measured from XOX' by the symbol y, and $ON = b$, the equation $y = b$ is true for all points on RNP.

At the point P, at which these lines intersect, $x = a$, $y = b$.

3. If, however, we take points M', N' on OX', OY', such that $OM' = OM$, $ON' = ON$, and through M', N' draw straight lines parallel to YOY', XOX' respectively, it would seem that at each of the points P, Q, R, S, where these four lines intersect, $x = a$, $y = b$: it is necessary therefore to adopt some convention or rule to distinguish between these four points.



Let us suppose that all lines drawn in any definite direction are considered positive; then those drawn in the opposite direction will be negative.

Let all lines drawn to the right of YOY' parallel to XOX' be positive, then those to the left will be negative; thus if $OM = a$, $OM' = -a$.

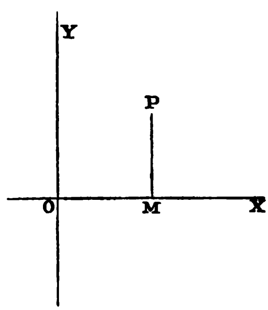
Similarly, let lines drawn parallel to YOY' above XOX' be positive, then those drawn below will be negative; thus if $ON = b$, $ON' = -b$.

If then, as before, x, y denote distance from YOY', XOX' respectively, then at P, $x = a$, $y = b$, at Q, $x = a$, $y = -b$, at R, $x = -a$, $y = b$, and at S, $x = -a$, $y = -b$.

4. *Axes, Coordinates, System of Coordinates.*

We can now explain what we mean by the coordinates of a point.

Let O be a fixed point in a plane XOY: let XOX', YOY' be two straight lines in that plane at right angles to each other, P any point in the plane.



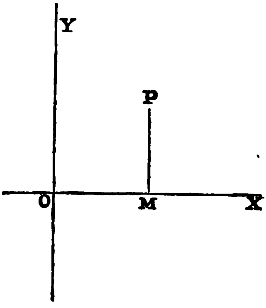
Through P draw PM perpendicular to OX, cutting it in M; then it is evident that if OM, PM are known in magnitude and direction or sign, the position of P is completely determined.

OM is obviously equal to the perpendicular from P on OY.

OM, PM are called the coordinates of P, and are denoted by x, y respectively: O is called the origin, OX the axis of x, since x is measured along it, OY the axis of y, since y is measured parallel to it.

OM, PM are said to be the coordinates of P belonging to the system XOY.

The point P, whose coordinates are x, y, is often called the point (xy): thus, a point, for which $x=a$, $y=b$, is called the point (ab).



When the point P is not completely determined, its coordinates are denoted by the variables x, y; x', y'; &c. When the position of a point is completely known, the coordinates are generally denoted by the letters a, b; h, k; or by x, y with suffixes such as x_1, y_1; x_2, y_2; &c.

Thus, if we want to determine the position of a point with reference to fixed points we shall use x, y for the coordinates of the *unknown* point, and a, b; h, k; x_1, y_1; &c., for the *known* coordinates of the fixed points.

These coordinates are called *Cartesian*, because the methods of using them were first given by Des Cartes, in the seventeenth century.

The student is now recommended to take a piece of paper, and draw two straight lines on it at right angles to each other, to measure off distances from the point of intersection along these lines equal to $\frac{1}{4}$ inch, and draw straight lines parallel to the original lines from these points, thus dividing the paper into small squares; he will then be able to do the following exercise.

EXAMPLES I. *a*.

1. Let 1 represent $\frac{1}{4}$ inch; indicate by a figure the relative positions of the following points: $(3, 1)$, $(2, 2)$, $(4, 5)$, $(3, -1)$, $(-1, 3)$, $(-3, -1)$, $(0, 2)$, $(0, -3)$, $(4, 0)$, $(0, 4)$, $(-2, 0)$, $(0, 0)$.

2. Take any lengths OA, OB along the axes of x and y respectively, let $OA=a$, $OB=b$; determine the position of the points $(0, a)$, $(0, b)$, $(a, 0)$, (a, b), $(-a, b)$, $(2a, -3b)$, $(-3a, 0)$, $\left(\frac{a}{2}, -b\right)$.

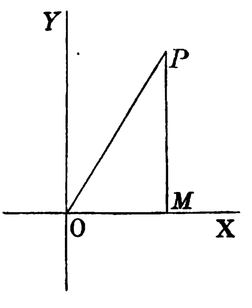
5. Distinction between an expression and an equation.

In Algebra, any determinate combination of letters is called an *expression*; a statement that two expressions are equal, or that one is zero, is called an *equation*.

So in Analytical Geometry we have *expressions* for the lengths of lines, areas of triangles, &c., and we obtain *equations* by putting these expressions equal to zero, or by making two of them equal.

We shall recur to these equations in the next chapter, but as we have constantly to use lengths of straight lines and areas in Geometry, it is necessary first to obtain the equivalent algebraic expressions.

6. *To find the distance of any point from the origin in terms of the coordinates of that point.*



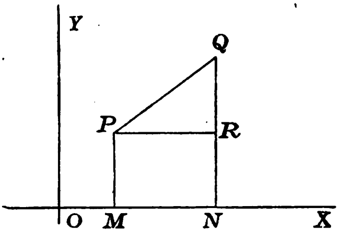
Let O be the origin, P the point whose distance OP is required; let the coordinates of P be x, y.

Then

$$OP^2 = OM^2 + PM^2 = x^2 + y^2, \text{ (Euc. i. 47.)}$$

$$\therefore\ OP = (x^2 + y^2)^{\frac{1}{2}}.$$

7. *To find the distance between two points.*



Let d be the distance required.

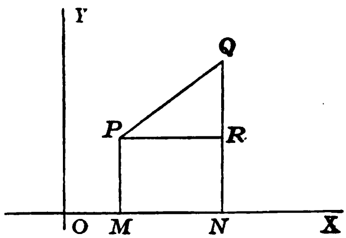
Let P, Q be the points, and let their coordinates be x_1, y_1; x_2, y_2, respectively.

Draw PR parallel to the axis of x cutting QN in R.

Then $$PQ^2 = PR^2 + QR^2.$$

But $\quad PR = MN = ON - OM = x_2 - x_1.$

Similarly,



$QR = QN - PM = y_2 - y_1;$

$\therefore PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$

or

$d = \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}^{\frac{1}{2}}.$

The student should draw figures in which P and Q are in different quadrants, so that any one or more of the quantities x_1, y_1, x_2, y_2 may be negative: he will find that the foregoing reasoning still holds good.

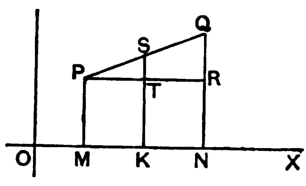
8. *To find the coordinates of the point which bisects the straight line joining two given points.*

Let S be the point required, OK, SK its coordinates: let SK cut PR in T.

Then $OK = OM + MK = OM + PT = OM + \frac{1}{2}PR$

$$= x_1 + \frac{x_2 - x_1}{2} = \frac{x_1 + x_2}{2}.$$

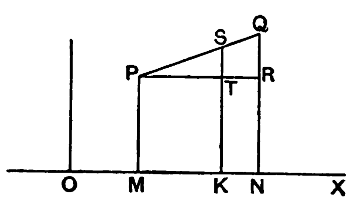
Similarly, $\quad SK = \dfrac{y_1 + y_2}{2}.$



Similarly we may find the coordinates of the point which divides the straight line joining two given points in a given ratio.

Let the given ratio be $m : n$, so that

$PS : SQ :: m : n.$



Then, with the same construction,

$PT : TR = m : n.$

$\therefore PT : PR = m : m + n.$

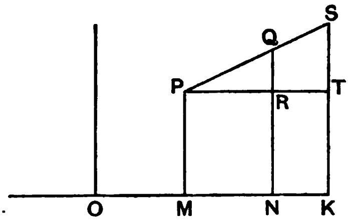
Hence $$OK = OM + PT = \frac{mx_2 + nx_1}{m+n}.$$

Similarly, $$SK = \frac{my_2 + ny_1}{m+n}.$$

Since we are only concerned with the ratio $m : n$, we may assume that $m + n = 1$, and this is usually a convenient assumption to make.

Suppose S to be in PQ produced so that $PS : SQ = m : n$.

Then since PS, SQ are drawn in opposite directions, they must be of opposite signs; writing $-n$ for n in the preceding proof it will be seen still to hold good.



The line is now said to be externally divided.

If S and S' are two points such that PQ is internally and externally divided in the same ratio, PQ is said to be divided harmonically, and S, S' are called conjugate points.

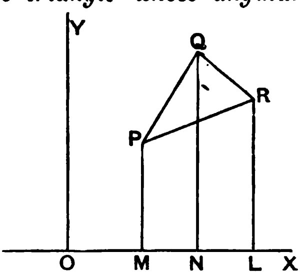
It is easy to see that $\frac{1}{PS} + \frac{1}{PS'} = \frac{2}{PQ}$, the condition that PS, PQ, PS' form a harmonic progression.

9. *To find the area of the triangle whose angular points are given.*

Let PQR be the triangle, and let the coordinates of P, Q, R be x_1y_1, x_2y_2, x_3y_3 respectively.

Then $\triangle\ PQR$

$= PQNM + RLNQ - PMLR.$

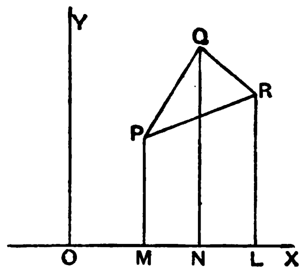


But $PQNM = \frac{1}{2}MN(PM + QN) = \frac{1}{2}(x_2 - x_1)(y_2 + y_1)$*.
Similarly, $RLNQ = \frac{1}{2}(x_3 - x_2)(y_3 + y_2)$,
$PMLR = \frac{1}{2}(x_3 - x_1)(y_3 + y_1)$;

$\therefore$ area required

$$= \tfrac{1}{2}\{(x_2 - x_1)(y_2 + y_1) + (x_3 - x_2)(y_3 + y_2) - (x_3 - x_1)(y_3 + y_1)\}$$
$$= \tfrac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1).$$

In this expression the coordinates must be taken in such an order as to make the whole expression positive.



The student should convince himself, by placing P, Q, R in different positions, that this value is still the same in magnitude, though with a possible change of sign; if for instance, we interchange the positions of P and Q, so that we interchange x_1 with x_2, y_1 with y_2, the whole expression will be changed in sign, and $\therefore$ its numerical value will be unaltered.

Again if Q is below PR we shall have the area changed in sign but not in value, and so too if any of the coordinates are negative.

Different cases are given in the following questions, in every one of which a figure should be drawn.

EXAMPLES I. *b*.

1. Determine the distances of the following points from the origin: (2, 3), (12, 5), (– 9, 40), ($3a$, $4a$), ($-2b$, b), ($a \sin \alpha$, $a \cos \alpha$).

2. Determine the distances between the following pairs of points: 3, 4 and 4, 3; – 3, 4 and 4, – 3; 1, 1 and – 1, – 1; 1, – 1 and – 1, – 1; – 1, 1 and 1, 1; 1, 1 and – 3, – 2; – 1, – 5 and 11, 0; h, k and $2h$, $-3k$; a, b and b, a; $-3a$, $2a$ and $-9a$, $-6a$.

* If this is not obvious, produce MP, NQ to M', N' so that $PM' = QN$, $QN' = PM$, then $MM' = NN' = y_1 + y_2$. $MM'N'N$ is a rectangle, of which $MPQN$ is half.

3. Find the coordinates of the middle points of the lines joining the pairs of points in question 2.

4. Find the coordinates of the points which divide the same lines internally and externally in the ratio 2 : 1.

5. Prove geometrically that the area of the triangle formed by joining the origin to the points (hk), $(h'k')$ is $\frac{1}{2}(hk' \sim h'k)$.

6. Determine the areas of the triangles whose angular points are respectively

(i) 3, 4; 0, 2; 3, 0. (ii) 1, 0; 4, 0; 0, 2. (iii) 0, 3; 0, -3; 2, 0. (iv) 3, 1; 1, 3; 1, 5. (v) 3, -1; -1, -3; 1, 5. (vi) -3, -1; 1, 3; -1, -5. (vii) -1, 3; 3, 1; 3, -5. (viii) a, b; $-a, -b$; h, k. (ix) a, b; $a, -b$; $-a$, 0. (x) $a\cos\alpha$, $a\sin\alpha$; $a\cos\beta$, $a\sin\beta$; 0, 0.

7. Find the area of the quadrilateral whose angular points are $(0, 0)$, (x_1y_1), (x_2y_2), (x_3y_3).

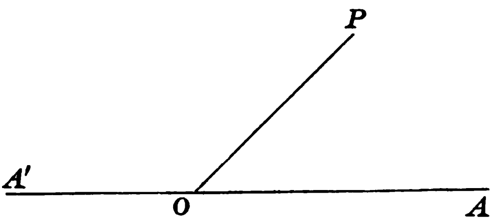
Divide the quadrilateral into two triangles.

8. Find the area of the quadrilateral whose angular points are (x_1y_1), (x_2y_2), (x_3y_3), (x_4y_4).

10. *Polar Coordinates.*

There is another system of coordinates which is often useful. Let O be, as before, a fixed point, AOA' a fixed straight line, P any point. Join OP.

It is evident that P is known in position if we know its distance from O, and the angle that distance makes with OA.



Thus, if we denote the distance OP by r, and the angle AOP by θ, the position of P is determined if r and θ are known.

r, θ are called the polar coordinates of P; O is called the *pole*, OA the *initial line*, OP the *radius vector of* P.

As in Trigonometry, the angle AOP is considered positive when measured in the direction opposite to that of the order of figures on a watch or counter clock-wise, negative when in that direction.

Examples I. *c*.

Indicate by a figure the relative position of the following points, when a represents $\frac{1}{2}$ inch;

$$(a,\ 0),\ \left(a,\ \frac{\pi}{2}\right),\ (2a,\ 30^\circ),\ \left(a\cos\frac{\pi}{3},\ \frac{\pi}{3}\right),\ \left(5a,\ \tan^{-1}\frac{4}{3}\right),$$

$$\left(-5a,\ \tan^{-1}\frac{3}{4}\right),\ (-a,\ 30^\circ),\ (a,\ 210^\circ).$$

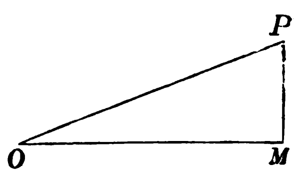
11. We can connect these two systems.

Let O be the same for both systems: let the axis of x be the initial line, and let the coordinates of P be x, y; r, θ referred to the two systems, respectively.

Then $OM = OP \cos POM$, $PM = OP \sin POM$,

or
$$x = r\cos\theta,\quad y = r\sin\theta.$$

It is easy to see that as θ passes through all values from 0 to 2π, the signs of x and y are the same as those of $\cos\theta$ and $\sin\theta$, respectively. Conversely, if we wish to obtain the polar coordinates from the rectangular, we have



$$r^2 = r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2,\quad \tan\theta = \frac{y}{x}.$$

Examples I. *d*.

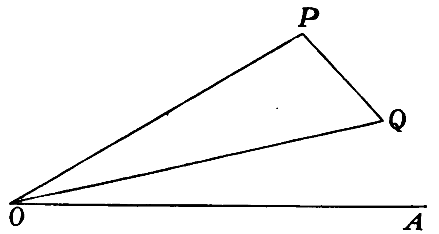
1. Change the coordinates from rectangular to polar in the equations

$$y = x,\ x^2 + y^2 = a^2,\ x^2 - y^2 = a^2,\ y^2 = 4ax,\ x\cos\alpha + y\sin\alpha = p.$$

2. Change the coordinates from polar to rectangular in the equations

$\theta = \frac{3\pi}{4}$, $r = c$, $\frac{l}{r} = \cos\theta + \sin\theta$, $r^2 \sin 2\theta = a^2$, $r = a\cos\theta$, $r\cos\theta = a$.

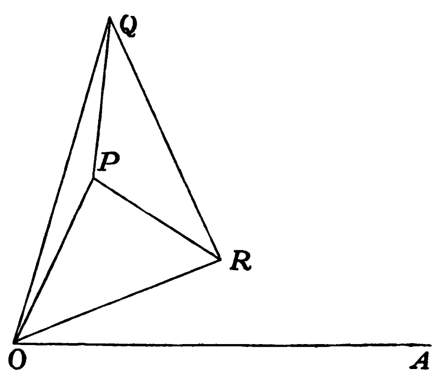
12. *To find the distance between two points in polar coordinates.*



Let $OP = r_1,\ OQ = r_2,\ POA = \theta_1,\ QOA = \theta_2,$

then
$$PQ^2 = OP^2 + OQ^2 - 2OP\,.\,OQ\cos POQ$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 \sim \theta_2).$$

13. *To find the area of a triangle in polar coordinates.*



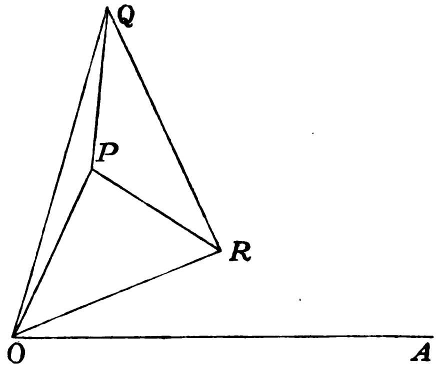
Let

$OP = r_1,\ OQ = r_2,\ OR = r_3,\ POA = \theta_1,\ QOA = \theta_2,\ ROA = \theta_3;$

then $$\triangle PQR = \triangle QOR - \triangle QOP - \triangle POR$$

$$= \tfrac{1}{2}\{r_2 r_3 \sin(\theta_2 - \theta_3) - r_2 r_1 \sin(\theta_2 - \theta_1) - r_1 r_3 \sin(\theta_1 - \theta_3)\}$$

$$= \tfrac{1}{2}\{r_2 r_3 \sin(\theta_2 - \theta_3) + r_1 r_2 \sin(\theta_1 - \theta_2) + r_3 r_1 \sin(\theta_3 - \theta_1)\}.$$



This may be written in the form,

$$\tfrac{1}{2} r_1 r_2 r_3 \left\{\frac{\sin(\theta_2 - \theta_3)}{r_1} + \frac{\sin(\theta_3 - \theta_1)}{r_2} + \frac{\sin(\theta_1 - \theta_2)}{r_3}\right\}.$$

In this expression the coordinates must be taken in such an order as to make the whole expression positive.

Examples I. *e*.

1. A regular octagon is inscribed in a circle of radius a; determine the coordinates of its angular points, taking the centre of the circle as origin, and two diameters passing through four angular points as axes.

2. Tangents are drawn to the circle at the vertices of the octagon in the preceding question: determine the coordinates of the vertices of the octagon so formed.

3. Determine the coordinates of the vertices of a regular hexagon the length of one side being a, one angular point being origin, and a side the axis of x.

4. ABC is a triangle, S, I the centres of the circumscribed and inscribed circles; find the coordinates of S, I (1) taking A as origin and AB as axis of x, (2) taking A as pole and AB as the initial line.

5. The coordinates of the angular points A, B, C of a triangle are (h_1k_1), (h_2k_2), (h_3k_3) respectively; D, E, F are the middle points of BC, CA, AB respectively: determine the coordinates of D, E, F.

6. In the preceding question, if DA be divided in G, so that $DG : GA :: 1 : 2$, determine the coordinates of G, and hence shew that AD, BE, CF are concurrent (meet in a point).

7. If R be the middle point of PQ in Art. 12, r, θ its coordinates, find r, θ in terms of r_1, θ_1, r_2, θ_2.

8. Find the areas, in polar coordinates, of the following triangles having given the angular points:

(1) The pole, $(a, 0)$, $\left(b, \frac{\pi}{2}\right)$. (2) The pole, $(a, 0)$, $\left(a, \frac{\pi}{3}\right)$.

(3) $(a, 0)$, $\left(2a, \frac{\pi}{3}\right)$, $\left(3a, \frac{2\pi}{3}\right)$. (4) $\left(a, \frac{\pi}{3}\right)$, $\left(a, \frac{\pi}{2}\right)$, $\left(a, \frac{2\pi}{3}\right)$.

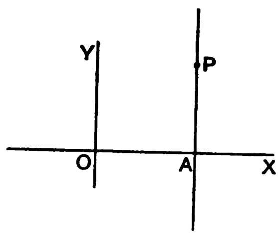
(5) $\left(a, \frac{\pi}{6}\right)$, $\left(2a, \frac{\pi}{6}\right)$, $(\sqrt{3}a, 0)$.

CHAPTER II.

THE STRAIGHT LINE. EQUATIONS.

14. CONSIDER the equation $x = a$.

Measure a length OA equal to a along the axis of x, and draw AP perpendicular to OA.



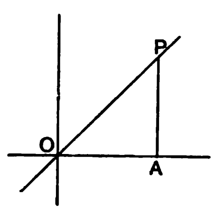
Then, if P be any point on this straight line, its coordinates are OA, AP; but $OA = a$; therefore $x = a$ represents the straight line AP.

Notice that all points for which $x = a$ must be on this line, and that there are no points on the line for which x has any value except a.

The equation therefore represents the line completely.

The line is said to be the *locus of the equation,* and the equation is said to be the *equation to the line.*

Just in the same way the equation $y = b$ represents a straight line parallel to the axis of x and at a distance b from it. So, in polar coordinates $r = c$ is true for all points whose distance from the pole is c, and for no others, its locus therefore is the circle whose centre is the pole and radius c.



Again, take the equation $x = y$.

Measure any length OA along the axis of x and draw AP equal to OA and perpendicular to it.

Then $POA = 45°$.

Therefore $x = y$ represents the straight line through O which makes an angle 45° with the axis.

Next, take the indeterminate equation of the first degree

$$Ax + By + C = 0.$$

Here we may give any value we please to one of the quantities, x or y, but if once we assign any particular value to either we thereby fix the other.

The equation therefore represents an infinite number of points, the coordinates of each of which when substituted in the equation $Ax + By + C = 0$ make it identically true. These points form a line, and this line is called the *locus of the equation*, and conversely the equation which is true for all points of the line is called the *equation to the line.*

Conversely, if we draw a straight line or a curve, at each point on it there must be some definite relation between its coordinates, that relation will be expressed by means of an equation, which is called the *equation to the line*, and the line is said to be the *locus of the equation.*

If the equation given is quadratic, it is evident that for each definite value of one variable, we shall, in general, get two values of the other.

15. *To trace the form of a curve represented by any equation.*

Take, as before, paper ruled in squares, and give x a series of values -2, -1, 0, 1, 2 etc. in succession.

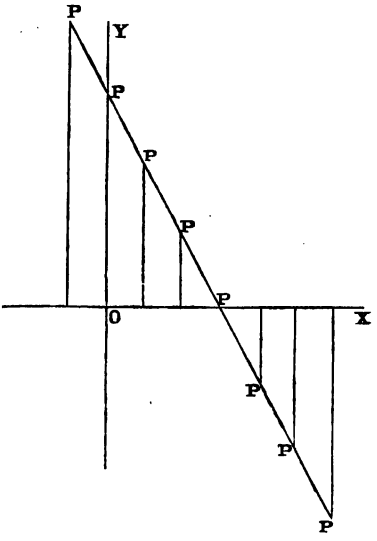
Then the values of y can be written down at once, and so a series of points on the locus determined.

By taking a considerable number of such points, we can trace the shape of the locus.

Of course, it will do equally well to assign a set of values to y, and obtain the corresponding values of x. We will take some examples.

(1) $2x + y = 6$.

Give x the values -1, 0, 1, 2, 3, 4, 5, 6, then we get for y, 8, 6, 4, 2, 0, -2, -4, -6. Hence by drawing lines as in the figure the points marked P lie on a straight line.



(2) $x^2 + y^2 = 16$.

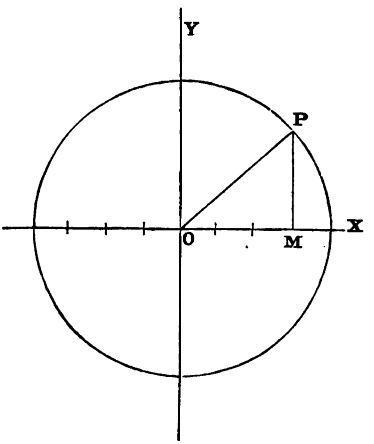
Here, if either x^2 or $y^2 > 16$, the other variable becomes impossible: this shews that the whole figure lies within the square formed by the four lines $x = 4$, $y = 4$, $x = -4$, $y = -4$.

Again, if P be a point in the figure

$$OM^2 + PM^2 = OP^2;$$

$$\therefore OP^2 = 16, \quad OP = 4.$$

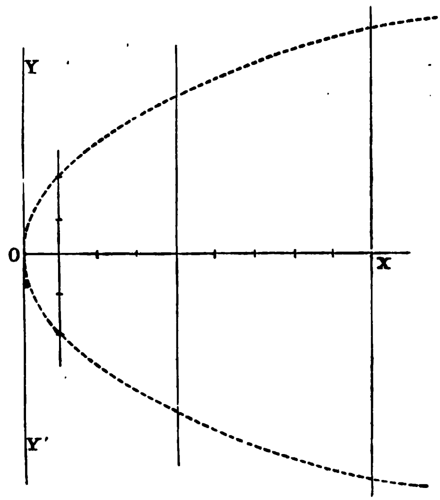
This shews that P must be a point on the circle whose centre is O and radius 4.



(3) $y^2 = 4x$.

Here if x is negative y is impossible, therefore no part of the locus lies to the left of YOY'.

For every value of x there are two equal and opposite values of y; the locus is therefore a curve such that it is divided into two exactly equal parts by the axis of x.



Now give x the values 0, 1, 4, 9, 16, &c. in succession, then the values of y are

$0, \pm 2, \pm 4, \pm 6, \pm 8$, &c.

As x increases y increases without limit.

The dotted line in the figure will therefore represent the curve.

We have given two examples of quadratic equations, but the student should confine his attention for the present to simple equations.

The following questions should be worked through very carefully, giving x or y a series of values, determining the corresponding values of y or x, and marking down the position of the points thus found.

If one of the variables has unity for its coefficient, it is easier to trace the locus by assigning values to the other, thus in the equation $3x+y=4$, we should give values $-1, 0, 1$ in succession to x, but in $x-3y=4$ to y.

The student should not use a separate diagram for each equation: he will get a much better notion of the way in which equations represent lines, if he takes the same axes for at least six.

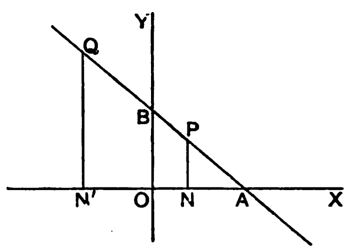
EXAMPLES II. *a*.

1. $x = 3y$.	2. $3x + y = 0$.	3. $x - 3y = 4$.
4. $x + 3y + 4 = 0$.	5. $3x + y = 4$.	6. $2x + 3y = 9$.
7. $3x - 2y = 1$.	8. $3x + 4y = 12$.	9. $x + y = 6$.
10. $x - y = 1$.	11. $4x + 5y + 1 = 0$.	12. $5x - 4y = 1$.
13. $3x = 5y + 6$.	14. $2y = 8x - 1$.	15. $\frac{x}{3} + \frac{y}{2} = 1$.
16. $\frac{x}{2} - \frac{y}{3} = 1$.	17. $\frac{x}{3} = \frac{y+3}{4}$.	18. $\frac{x}{2} + \frac{y-3}{4} = 0$.
19. $y = (x - 2) \tan 30°$.	20. $y - 2 = x \tan 60°$.	

The student should notice, that in every one of the preceding equations, the points obtained lie on a straight line: if he has done this, he will be able to understand the equation to a straight line under various forms, which it is the object of this chapter to discuss.

16. *To find the equation to the straight line which passes through two fixed points on the axes.*

Let the straight line cut the axis of x in the point A and that of y in the point B; let $OA = a$, $OB = b$; let the coordinates of P, any point in the line, be x and y.



Then,

$\because PN$ is parallel to BO,

$$\therefore \frac{PN}{BO} = \frac{AN}{AO} = \frac{AO - ON}{AO},$$

or,

$$\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a},$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1.$$

This equation is therefore satisfied by every point in AB, and by no other points, since it is deduced from the geometrical relation $PN : OB :: AN : OA$, which is only true when P lies on AB; it is therefore the equation to AB.

It may easily be seen that this equation holds for all points on the line AB, or AB produced either way.

For let Q be such a point; then

$$\frac{QN'}{BO} = \frac{AN'}{AO} = 1 + \frac{ON'}{AO},$$

$$\therefore \frac{-ON'}{AO} + \frac{QN'}{BO} = 1,$$

but if x, y, be the coordinates of Q, $x = -ON'$, $y = QN'$,

$$\therefore \frac{x}{a} + \frac{y}{b} = 1, \text{ as before.}$$

17. Conditions that two equations shall represent the same locus.

Suppose that we have two equations, one that to a known line, another which we wish to interpret, we can find the conditions that the two equations shall represent the same locus.

In general, two equations taken together will give determinate values of x and y, and therefore represent points; if, however, the coefficient of every term in the second is the same multiple of the coefficient of the corresponding term of the first, we may divide by that constant multiple and so obtain that first equation.

The condition required is therefore that the ratio of the coefficients of x^2, xy, y^2, x, y, &c. and of the constant terms in the two equations should be the same.

In practice, it is best to divide each equation by the coefficient of some term or by the constant term; then the coefficients of the other terms must be identical.

This process is called equating coefficients.

18. To prove that the general equation of the first degree $lx + my = d$ always represents a straight line.

The equation $lx + my = d$ may be written

$$\frac{lx}{d} + \frac{my}{d} = 1.$$

Now the equation to the straight line which cuts off intercepts a, b from the axes is

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ (Art. 16).}$$

Hence, these equations will represent the same line if

$$\frac{l}{d} = \frac{1}{a}, \quad \frac{m}{d} = \frac{1}{b}, \text{ or } \frac{d}{l} = a, \quad \frac{d}{m} = b.$$

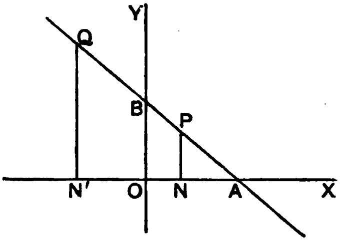
Now since we can always measure off lengths equal to $\frac{d}{l}$, $\frac{d}{m}$ along the axes, the equation $lx + my = d$ must represent the straight line which cuts off intercepts $\frac{d}{l}$, $\frac{d}{m}$ from the axes of x and y respectively.

19. This proposition is so important that we will prove it independently.

Let $lx + my = d$ be an equation, and P any point whose coordinates (x, y) satisfy the equation so that

$$l \,.\, ON + m \,.\, PN = d,$$

then P shall lie on a certain straight line.



Measure $OA = \frac{d}{l}$, $OB = \frac{d}{m}$, and join AP, BP.

Now $$PN = y = \frac{d - lx}{m},$$

and $$AN = OA - ON = \frac{d}{l} - x = \frac{d - lx}{l};$$

$$\therefore \frac{PN}{AN} = \frac{l}{m} = \frac{\frac{d}{m}}{\frac{d}{l}} = \frac{OB}{OA};$$

$\therefore$ P is a point on the straight line AB.

This equation is often written $Ax + By + C = 0$.

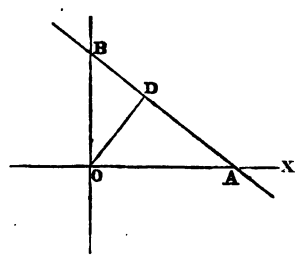
In this case the intercepts on the axes are $-\frac{C}{A}$, $-\frac{C}{B}$ respectively.

Since there are three constants in this equation, it would seem that a straight line could satisfy three conditions: this, however, is not the case, since we can divide by any one of the three without altering the equation, and then the straight line which it represents will be completely determined by the ratios thus obtained.

20. *To find the angle which the line*

$$lx + my = d$$

makes with the axis of x, and the perpendicular on it from the origin.



Let $OA = \frac{d}{l}$, $OB = \frac{d}{m}$, then, by the preceding Article, AB is the line required.

Let BAX, the angle which BA makes with the positive direction of the axis of x, be called θ.

Draw OD perpendicular to AB, and let $OD = \delta$.

$$\text{Then } \tan\theta = -\tan BAO = -\frac{OB}{OA} = -\frac{\frac{d}{m}}{\frac{d}{l}} = -\frac{l}{m}.$$

Again $AB.OD = OB.OA$, since each of these rectangles is double the triangle OAB.

But $AB^2 = OA^2 + OB^2 = \frac{d^2}{l^2} + \frac{d^2}{m^2} = \frac{d^2}{l^2m^2}(l^2 + m^2)$;

$$\therefore \frac{d\delta(l^2+m^2)^{\frac{1}{2}}}{lm} = \frac{d^2}{lm}, \quad \delta = \frac{d}{(l^2+m^2)^{\frac{1}{2}}}.$$

This perpendicular may always be considered to be positive, provided we regard the angle it makes with the axis of x as having any magnitude less than 360°.

21. We are now in a position to interpret any simple equation, and to draw the straight line which it represents.

Put $y = 0$ in the equation, the corresponding value of x will give the point at which the line cuts the axis of x.

Put $x = 0$, we obtain the point at which it cuts the axis of y.

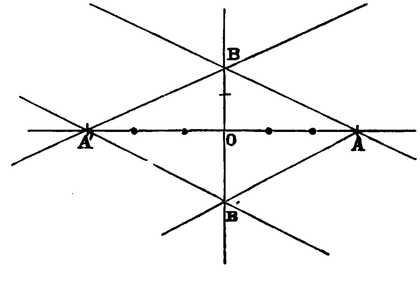
Join these points, and we obtain the line required.

We may observe that if the coefficients of x and y have the same sign, the line makes an obtuse angle with the axis of x, if different, an acute.

EXAMPLES. Interpret the equations:

$$x + 2y = 3; \qquad x + 2y + 3 = 0;$$
$$x - 2y = 3; \qquad x - 2y + 3 = 0.$$

In the first equation put $y = 0$, $\therefore x = 3$; then put $x = 0$, $\therefore y = \frac{3}{2}$.



Now measure $OA = 3$ along the axis of x, and $OB = \frac{3}{2}$ along the axis of y. AB is the line required.

To interpret

$x + 2y + 3 = 0$:

this line passes through the points $x = -3$, $y = 0$; $x = 0$, $y = -\frac{3}{2}$; measure $\therefore$ $OA' = OA$ and $OB' = OB$ along the negative parts of the axes. $A'B'$ is the line required.

So the third and fourth equations represent AB', $A'B$ respectively.

To determine the angles which these lines make with the axes, divide the coefficient of x by that of y and change the sign; then $-\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ are the tangents of these angles.

The perpendicular from the origin is

$$\frac{3}{\sqrt{5}} \text{ in each case.}$$

EXAMPLES II. *b*.

1. Draw the straight lines represented by the following equations, where 1 represents a line $\frac{1}{2}$ an inch in length.

$$x = 2,\ x + 3 = 0,\ y = 1,\ y + 4 = 0,\ x = y,\ 2x + 3y = 0.$$

$$4x + 3y = 5,\ 4x - 3y = 5,\ 4x - 3y + 5 = 0,\ 4x + 3y + 5 = 0.$$

$$\frac{x}{2} + \frac{y}{3} = 1,\ \frac{x}{3} + \frac{y}{2} = 1,\ \frac{x}{2} - \frac{y}{3} = 1,\ \frac{x}{2} + \frac{y}{3} + 1 = 0,\ \frac{y}{3} - \frac{x}{2} = 1.$$

2. Determine the angle which each of the above lines makes with the axis of x, and the perpendicular from the origin.

3. If θ be the angle which the perpendicular from the origin on $lx + my = d$ makes with the axis of x, where the perpendicular is supposed to have revolved counterclockwise from the position in which $\theta = 0$,

$$\cos\theta = \frac{l}{\sqrt{l^2 + m^2}},\ \sin\theta = \frac{m}{\sqrt{l^2 + m^2}},\ d \text{ being always positive.}$$

4. Find the value of θ in the case of each of the lines in question 1, in which both x and y enter the equation.

22. To find the equation to the straight line which passes through the points x_1, y_1; x_2, y_2.

Let the coordinates of any point on this line be x, y.

Then the area of the triangle whose angular points are x, y; x_1, y_1; x_2, y_2 must be zero.

$\therefore$ (Art. 14) $xy_1 - x_1y + x_1y_2 - x_2y_1 + x_2y - xy_2 = 0$.

Otherwise, suppose $lx + my = d$ to be the equation required.

$$\therefore lx + my = d \quad\text{.......... (1)},$$

$$lx_1 + my_1 = d \quad\text{.......... (2)},$$

$$lx_2 + my_2 = d \quad\text{.......... (3)}.$$

Subtract (2) from (1) $\therefore l(x - x_1) + m(y - y_1) = 0$,

$$\therefore \frac{l}{m} = -\frac{y - y_1}{x - x_1},$$

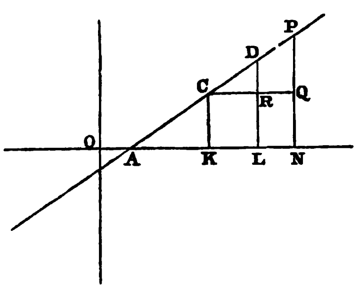
subtract (2) from (3) $\therefore l(x_2 - x_1) + m(y_2 - y_1) = 0$,

$$\frac{l}{m} = -\frac{y_2 - y_1}{x_2 - x_1},$$

$\therefore \dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$ is the equation required. This is more conveniently written $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

We can obtain this geometrically very easily.

Let C, D be the points (x_1y_1), (x_2y_2), P any point (xy) on CD produced either way.



Draw CK, DL, PN perpendicular to the axis of x and CRQ perpendicular to PN cutting DL, PN in R, Q respectively.

Then

$$OK = x_1, \quad OL = x_2, \quad ON = x;$$
$$CK = y_1, \quad DL = y_2, \quad PN = y.$$

$\therefore\ CR = x_2 - x_1,\ DR = y_2 - y_1,\ CQ = x - x_1,\ PQ = y - y_1$

Now by similar triangles CDR, CPQ, $\dfrac{PQ}{DR} = \dfrac{CQ}{CR}$;

that is, $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$, the equation required.

This equation may be written in either of the following forms:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$
$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1),$$
$$y(x_2 - x_1) - x(y_2 - y_1) = x_2 y_1 - x_1 y_2,$$
$$xy_1 - x_1 y + x_2 y - xy_2 + x_1 y_2 - x_2 y_1 = 0.$$

This result may also be obtained from Art. 13, by supposing P to divide CD in the ratio $m : n$, putting

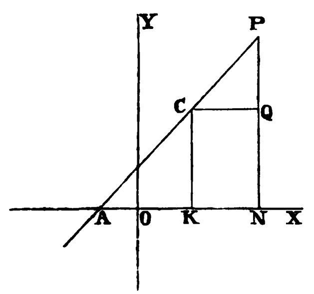
$$x = \frac{mx_2 + nx_1}{m + n},\ y = \frac{my_2 + ny_1}{m + n},$$

and eliminating the indeterminate quantity $\dfrac{m}{n}$.

23. *To find the equation to a straight line which passes through a given point, and makes a given angle with the axis of x.*

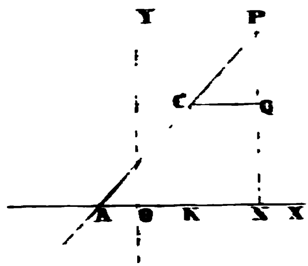
Let ACP be the straight line, C the given point $(x_1 y_1)$, P the point (xy), and let $CAX = \alpha$.

Draw CQ parallel to OX cutting PN in Q.



Then, in the figure,

$$ON = x,\ PN = y,\ OK = x_1,$$
$$CK = y_1.$$
$$\therefore\ CQ = x - x_1,\ PQ = y - y_1.$$



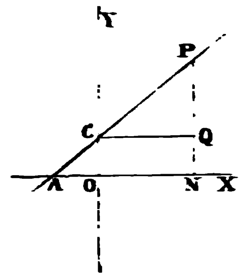
But

$$\tan \alpha = \frac{PQ}{CQ} = \frac{y - y_1}{x - x_1}.$$

$$\therefore \quad y - y_1 = (x - x_1) \tan \alpha$$

is the equation required.

If $\tan \alpha = \mu$, we see that $y - y_1 = \mu (x - x_1)$ represents a straight line which passes through $x_1 y_1$, and makes an angle $\tan^{-1} \mu$ with the axis of x.



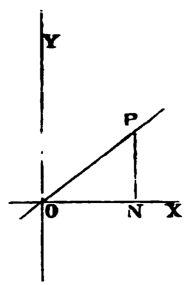
Two special cases should be noticed.

(1) Let C be on the axis of y: then $x_1 = 0$.

$$\therefore y - y_1 = \mu x$$

represents the straight line.

This equation is often written $y = mx + c$, where c is the intercept on the axis of y.



(2) Let C coincide with O, and therefore Q with N.

Then the equation becomes

$$y = \mu x.$$

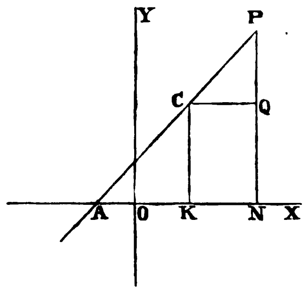
These forms can be easily proved independently.

24. The equation

$$y - y_1 = (x - x_1) \tan \alpha$$

may be connected with polar coordinates and expressed in a very convenient form.

Let CP, the distance between the fixed point and any point on the line, be denoted by r;



$\therefore\ CQ = r\cos\alpha,\ PQ = r\sin\alpha,$

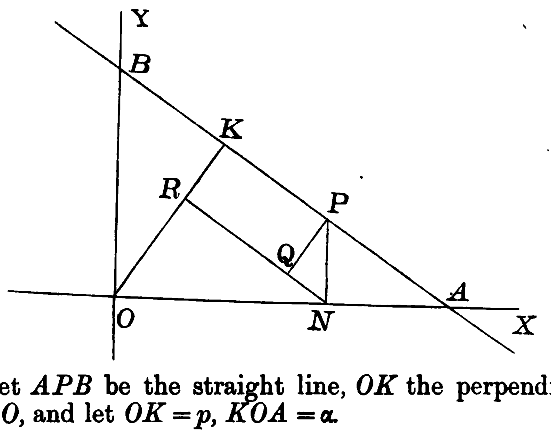
or

$x - x_1 = r\cos\alpha,\ y - y_1 = r\sin\alpha;$

$\therefore\ x = x_1 + r\cos\alpha,\ y = y_1 + r\sin\alpha.$

These equations are sometimes written in the form $\dfrac{x - x_1}{\cos\alpha} = \dfrac{y - y_1}{\sin\alpha} = r$, and are very useful when we want to determine the rectangle, square, &c. of lines through a fixed point, and know a curve on which the other end of the line lies.

25. *To find the equation to a straight line in terms of the perpendicular from the origin and the angles which that perpendicular makes with the axes.*



Let APB be the straight line, OK the perpendicular from O, and let $OK = p$, $KOA = \alpha$.

Draw NR parallel to AB, cutting OK in R, and draw PQ perpendicular to NR.

Then $$OR + RK = OK = p,$$

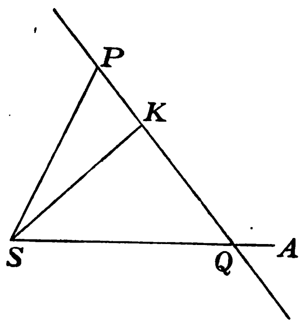
but $$OR = ON\cos\alpha = x\cos\alpha,$$

and $$RK = PQ = PN \sin \alpha = y \sin \alpha,$$
since $$QNP = RON = \alpha;$$
$$\therefore x \cos \alpha + y \sin \alpha = p,$$
the equation required.

26. *To find the polar equation to a straight line.*

Let PQ be the line required, P any point (r, θ) on it.

Let the perpendicular SK from the pole S make an angle $KSQ = \alpha$ with the initial line.



Then $SP \cos PSK = SK$,

or $$r \cos (\theta - \alpha) = p,$$
the equation required.

This might also be obtained by writing $r \cos \theta$, $r \sin \theta$ for x and y respectively in the equation $x \cos \alpha + y \sin \alpha = p$.

The general polar equation to a straight line may be obtained from the rectangular equation
$$lx + my = d,$$
by writing $r \cos \theta$, $r \sin \theta$ for x and y respectively.

It becomes $$lr \cos \theta + mr \sin \theta = d,$$
or $$l \cos \theta + m \sin \theta = \frac{d}{r}.$$

27. *Polar equation to the straight line through two fixed points.*

Let $(r_1, \theta_1)(r_2, \theta_2)$ be the coordinates of the fixed points, (r, θ) of any other point.

Then the area of the triangle of which these points are the vertices is
$$\frac{r_1 r_2 r}{2}\left(\frac{\sin(\theta_1 - \theta_2)}{r} + \frac{\sin(\theta_2 - \theta)}{r_1} + \frac{\sin(\theta - \theta_1)}{r_2}\right): \text{ (Art. 13).}$$

Now, if (r, θ) lie on the straight line joining the two points (r_1, θ_1), (r_2, θ_2) this area must vanish.

Hence $$\frac{\sin(\theta_1 - \theta_2)}{r} + \frac{\sin(\theta_2 - \theta)}{r_1} + \frac{\sin(\theta - \theta_1)}{r_2} = 0$$

is the equation required.

This equation, if expanded, is easily shewn to be of the form

$$l \cos\theta + m \sin\theta = \frac{d}{r}.$$

EXAMPLES II. c.

1. Interpret the following equations, and draw the lines represented:

$$y - 3 = 2(x - 2),\ y + 3 = \frac{x-2}{2},\ y + 1 = \sqrt{3}(x + 3),$$

$$y - 2 = \frac{1}{\sqrt{3}}(x + 1),\ \frac{x-2}{\sqrt{3}} = \frac{y-2}{1} = \frac{r}{2},\ x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3} = 2,$$

$$x\sin\frac{\pi}{3} - y\cos\frac{\pi}{3} = 2.$$

2. Find the equations to the straight lines which pass through the following pairs of points: (a, b), (b, a) : (h, k), $(-h, -k)$: (h, k), $(h, -k)$: $(h, -k)$, $(-h, -k)$: $(3, 4)$, $(1, 2)$: $(5, 6)$, $(0, -1)$: $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$.

3. Find the equation to the straight line which passes through the point $(-1, 2)$ and makes an angle $30°$ with the axis of x.

4. Find the equation to the straight line which passes through the point $(2, 2)$ and makes an angle of $45°$ with the axis of x.

5. Find the equation to the straight line when the perpendicular from the origin makes an angle $-\alpha$ with the axis of x and its length is b.

6. If the equations $\frac{x}{a}+\frac{y}{b}=1$, $Ax+By+C=0$, $y=\mu x+\nu$, $x\cos\alpha+y\sin\alpha=p$, represent the same line, then $Aa=Bb$;

$$p=\frac{-C}{\sqrt{A^2+B^2}}=\frac{ab}{\sqrt{a^2+b^2}}=\frac{\nu}{\sqrt{1+\mu^2}};\ \mu\sin\alpha+\cos\alpha=0.$$

7. Find the length of the straight line drawn from the origin making an angle 45° with the axis of x to meet the line $x+y=2$.

8. Find the length of the line drawn from the point (2, 1) making an angle 30° with the axis to meet the line $y=\sqrt{3}x-2$.

9. Find the length of the line drawn from (hk) in a direction making an angle α with the axis of x to meet the line $y=mx+c$.

10. If the length of the line from (h, k) to meet $y=mx+c$ be a, what is the angle it makes with the axis of x?

11. The length of the line drawn from the origin to a point on $x\cos\alpha+y\sin\alpha=p$ being $p\sec\beta$, find the angle it makes with the axis of x.

12. Shew that the equations $r\cos\theta=a$, $r\sin\theta=a$ represent straight lines perpendicular and parallel to the initial line at a distance a from the pole.

13. Interpret the equations:

$$\theta=\frac{\pi}{3},\ \theta+\alpha=\pi,\ r\sin\left(\theta-\frac{\pi}{3}\right)=p,\ \frac{l}{r}=\cos\theta+\sin\theta.$$

14. Find the polar equations to the straight lines which pass through the following pairs of points:

$$a,\ 0;\ b,\ \frac{\pi}{2}:\ -a,\ 30^\circ;\ a,\ 60^\circ:\ a,\ 60^\circ;\ a,\ 120^\circ.$$

15. ABC is a triangle: if A be the pole, AB the initial line, find the equation to the straight lines through A, B, C respectively which are perpendicular to the opposite sides, and the coordinates of their point of intersection.

16. If $r\cos(\theta-\alpha)=p$, $l\cos\theta+m\sin\theta=\frac{d}{r}$, represent the same line

$$\frac{m}{l}=\tan\alpha,\ d=p\,(l^2+m^2)^{\frac{1}{2}}.$$

CHAPTER III.

THE STRAIGHT LINE, CONTINUED.

28. *To find the angle between two straight lines whose equations are given.*

Let the given equations be

$$lx + my = d,$$
$$l'x + m'y = d',$$

and let ϕ be the angle required.

Then ϕ is the difference of the angles made by these lines with any other straight line.

Now the tangents of the angles made by these straight lines with the axis of x are $-\frac{l}{m}$, $-\frac{l'}{m'}$, respectively. (Art. 20.)

$$\therefore \quad \tan\phi = \frac{-\frac{l}{m} + \frac{l'}{m'}}{1 + \frac{ll'}{mm'}} = \frac{lm' - l'm}{ll' + mm'}.$$

Considering ϕ to be the acute angle between the lines we must take the numerator as positive.

If the two straight lines are parallel, $\tan\phi = 0$.

For they make the same angle with the axis of x, that is, $\frac{l}{m} = \frac{l'}{m'}$.

If the two straight lines are at right angles $\tan\phi = \infty$; that is $ll' + mm' = 0$.

This may be seen independently, for if θ be the angle that $lx + my = d$ makes with the axis of x, θ' the angle which $l'x + m'y = d'$ makes with the same axis,

$$\cot\theta' = -\tan\theta \quad \text{or} \quad -\frac{m'}{l'} = \frac{l}{m}.$$

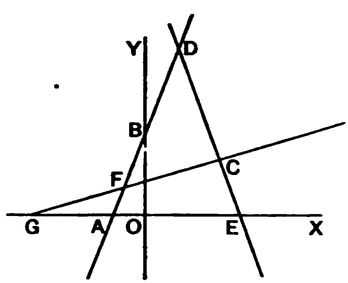
The equation $lx + my = d'$, where d' is indeterminate, represents a straight line parallel to $lx + my = d$, and by giving different values to d', may be made to represent any such line.

Similarly $mx - ly = d'$ represents any straight line perpendicular to $lx + my = d$.

It is obvious that the point of intersection of two straight lines is obtained by treating their equations as simultaneous.

29. *To find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.*

Let the equation to the given straight line AB, be reduced to the form $y = x \tan \alpha + c$, where $BAX = \alpha$.



Let C be the point $(x_1 y_1)$, then the equation

$$y - y_1 = m(x - x_1),$$

where m is indeterminate, may be made to represent any straight line through C. For it is the equation to some straight line which passes through C, since it is true when $x = x_1$, $y = y_1$, and any value can be given to m.

Now if

$$CDB = CFB = \beta, \quad CEX = \alpha + \beta, \quad CGX = \alpha - \beta,$$

$\therefore$ $y - y_1 = (x - x_1) \tan (\alpha \pm \beta)$ represents CD or CF as the upper or lower sign is taken.

Cor. $y - y_1 = (x - x_1) \tan \alpha$ represents the straight line through $x_1 y_1$ parallel to $y = x \tan \alpha + c$, and

$$(y - y_1) \tan \alpha + x - x_1 = 0$$

the perpendicular to it through the same point.

Thus $\frac{x}{a}+\frac{y}{b}=\lambda$, where λ is indeterminate, represents any parallel to $\frac{x}{a}+\frac{y}{b}=1$, and $ax-by=\lambda$ any perpendicular to the same line, while $\frac{x}{a}+\frac{y}{b}=\frac{x_1}{a}+\frac{y_1}{b}$, $ax-by=ax_1-by_1$ represent the parallel and perpendicular through x_1y_1.

Thus to obtain the equation to a straight line perpendicular to a given straight line we must interchange the coefficients of x and y and change the sign between them: this will give one side of the equation required, the other must be obtained from some other condition about the line.

Examples III, *a*.

1. Find the acute angles, or tangents of the acute angles, between the following pairs of straight lines, drawing a figure in each case.

(i) $x+y=3$, and $x=\sqrt{3}y$. (ii) $x=y$, and $y=2(x-1)$.

(iii) $y=2x-1$, and $x=2y-1$.

(iv) $\frac{x}{a}+\frac{y}{b}=1$, and $\frac{x}{b}+\frac{y}{a}=1$, where $a>b$.

(v) $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ and $y-y_1=\frac{y_2+y_1}{x_2+x_1}(x-x_1)$.

(vi) $x\cos\alpha+y\sin\alpha=p$, and $y-y_1=(x-x_1)\cot\alpha$.

2. Find the tangents of the *interior* angles of the triangles formed by the following sets of lines, verifying by the formula $\tan C=-\tan(A+B)$, and drawing figures.

(i) $y=0$, $y=2x$, $y=x+1$.

(ii) $y=0$, $\frac{x}{2}+\frac{y}{3}=1$, $y=2x+3$.

(iii) $x=0$, $2y=x+1$, $y=2(x+1)$.

(iv) $y=x$, $2y=x+2$, $y=3x-3$.

(v) $x+2y=3$, $2x+y=3$, $x+y=3$.

(vi) $(x-a)\cos\alpha+y\sin\alpha=0$, $y-b=x\tan\alpha$, $y+2(x-a)=0$.

3. Find the equation to the straight line which is parallel to $y = 2x - 3$, and passes through the point $(1, -2)$.

4. Find the equation to the straight line which is parallel to $3x + 4y = 5$, and passes through the point $8, -5$.

5. Find the equation to the straight line which is parallel to $\frac{x}{3} + \frac{y}{4} = 1$ and passes through the point $(-6, -8)$.

6. Find the equations to the straight lines which are parallel to $Ax + By + C = 0$, and pass through the following points respectively, $(10, 0)$, (a, b), $(Bc, -Ac)$.

7. Find the straight lines which are parallel to $y = mx + c$ and at a distance a from the origin.

8. Find the equation to the straight line which is perpendicular to $7x - 8y = 15$, and passes through the point $(1, 1)$.

9. Find the equation to the straight line which is perpendicular to $3x + 2y = 3$, and passes through the intersection of $7x - y = 10$, and $x + y = 6$.

10. Find the equations to two straight lines, which form right-angled triangles with $y = 2x$, and $2y = x$, and pass through the point $(1, 2)$.

11. Find the equation to the straight line which is perpendicular to $Ax + By + C = 0$, and cuts off a length b from the axis of y.

12. Find the equation to the straight line which is perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$, and passes through the point (a, b).

13. Find the equation to the straight line which passes through the point (b, a), and is perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$.

14. Find the equation to the straight line which is at a distance q from the origin, cuts the axis of y on the positive side, and is perpendicular to $x \cos \alpha + y \sin \alpha = p$.

15. Find the equations to the straight lines which pass through the origin, and make angles of 15° with $x + y = 2$.

16. Find the equations to the straight lines which pass through the origin and make an equilateral triangle with $y = \sqrt{3}x + 2$.

17. Find the equations to the straight lines which pass through $(a, 0)$ and make angles $\frac{\pi}{4}$ with $\frac{x}{a} + \frac{y}{b} = 1$.

18. Find the equations to the straight lines which cut off a length b from the axis of y and make angles β with

$$x \cos \alpha + y \sin \alpha = p.$$

30. *To find the perpendicular distance of a given point from a given straight line.*

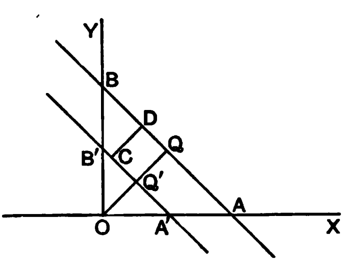


Fig. 1.

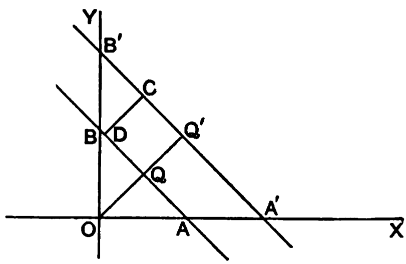


Fig. 2.

Let C be the given point, AB the given straight line, draw CD perpendicular to AB.

Through C draw $A'CB'$ parallel to AB cutting the axes in $A'B'$ respectively. Draw $OQ'Q$ perpendicular to AB and therefore perpendicular to $A'B'$.

Let the coordinates of C be h, k and the equation to AB be

$lx + my = d, \quad \therefore \; OQ = \frac{d}{(l^2 + m^2)^{\frac{1}{2}}}$ and is positive. (Art. 20.)

Now $lx + my = lh + mk$ is the equation to $A'B'$ (Art. 29)

$$\therefore \; OQ' = \frac{lh + mk}{(l^2 + m^2)^{\frac{1}{2}}}.$$

$$\therefore \; CD = OQ - OQ' = \frac{d - lh - mk}{(l^2 + m^2)^{\frac{1}{2}}}.$$

It is necessary to pay attention to the signs in this expression.

We have seen that the sign of a straight line denotes the direction in which it is measured, and have agreed (Art. 20) to consider the perpendicular from the origin on *any* straight line to be positive.

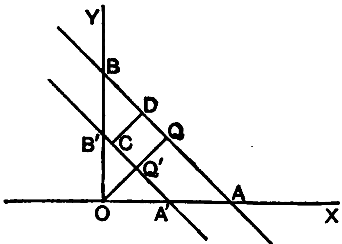


Fig. 1.

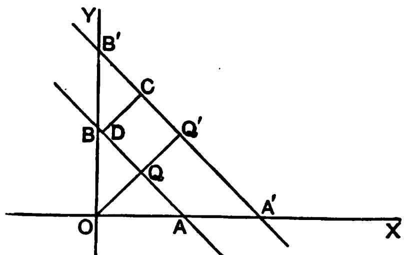


Fig. 2.

Hence the sign of the perpendicular from a point on a straight line should be positive if that point is the origin, that is if h and k vanish.

The expression above therefore gives the correct sign, provided d is positive.

If $lh + mk > d$ the perpendicular becomes negative; this shews that the point (hk) and the origin are on opposite sides of the line.

In fig. (1) CD is positive, in fig. (2) negative.

Notice that the perpendicular from $x_1 y_1$ on

$$x \cos \alpha + y \sin \alpha = p \text{ is } p - x_1 \cos \alpha - y_1 \sin \alpha.$$

31. *To find the distance between two parallel straight lines.*

The equations to the lines may be written in the form

$$lx + my = d,$$

$$lx + my = d',$$

where l and m are the same in sign and magnitude for both lines.

It is clear that the distance between the lines is the difference of their distances from the origin, that is $\dfrac{d-d'}{(l^2+m^2)^{\frac{1}{2}}}$.

Since we are only concerned with the actual length of this distance, we must consider this expression to be positive.

If d and d' are of the same sign, the lines are on the same side of the origin, if not the origin lies between them.

Definition. If we have any equation to a line written in the form $S=0$, where S is the result of taking all the terms to the left-hand side, and if we substitute the coordinates of any point in S, then the expression so obtained is called the *power of the point* with respect to the locus or equation.

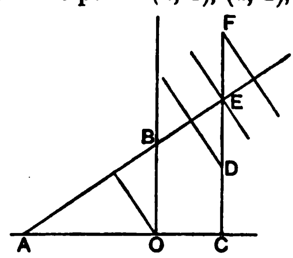
Thus $lh+mk-d$ is the power of hk with respect to $lx+my-d=0$.

We see then that the power of a point with respect to a line is proportional to the perpendicular from the point on the line.

If the power vanishes, it is clear that the point is on the line.

Examples worked out.

1. Find the perpendiculars from the points (1, 1), (1, 2), (1, 3), on the straight line whose equation is $3y-2x=4$: find also the equation to these perpendiculars, and their points of intersection with the line.

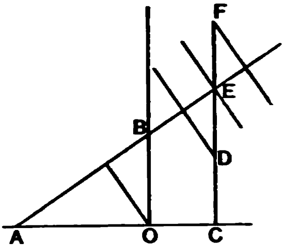


Divide both sides of the equation

$3y-2x=4$ by $\sqrt{3^2+2^2}$ or $\sqrt{13}$

the equation may now be written

$$\frac{4 + 2x - 3y}{\sqrt{13}} = 0,$$



$\therefore$ the perpendiculars required are $\dfrac{4+2-3}{\sqrt{13}}$, $\dfrac{4+2-6}{\sqrt{13}}$, $\dfrac{4+2-9}{\sqrt{13}}$, or $\dfrac{3}{\sqrt{13}}$, 0, $\dfrac{-3}{\sqrt{13}}$ respectively.

These results shew that the points are on the same side of the line as the origin, on the line, and on the side of it opposite to the origin, respectively.

The equations to the lines perpendicular to $3y - 2x = 4$ through the given points are (Art. 28)

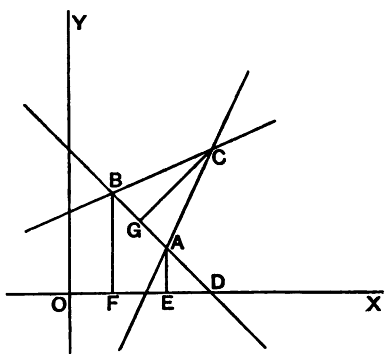
$3(x-1)+2(y-1)=0$, $3(x-1)+2(y-2)=0$, $3(x-1)+2(y-3)=0$, or $3x + 2y = 5$, $3x + 2y = 7$, $13x + 2y = 9$, respectively.

The coordinates of the feet of the perpendiculars are obtained by combining these equations with that to the original line: they are $(\frac{7}{13}, \frac{22}{13})$, $(1, 2)$, $(\frac{19}{13}, \frac{30}{13})$ respectively.

In the figure AB represents the original line, obtained by making $OA = -2$, $OB = \frac{3}{2}$, D, E, F the three points, obtained by measuring OC along the axis of x equal to unity, drawing CD perpendicular to OC, and giving the assigned values to CD, CE, CF.

2. To find the area of the triangle, when the equations to the sides are

$$x + y = 3 \ldots\ldots \text{(i)},$$
$$2x - y = 3 \ldots\ldots \text{(ii)},$$
$$2y - x = 3 \ldots\ldots \text{(iii)}.$$



Let AB, AC, BC be the three lines, respectively, drawn as in previous cases, then taking these equations in pairs simultaneously, it is easy to see that the coordinates of A, B, C are $(2, 1)$, $(1, 2)$, $(3, 3)$ respectively.

Now we could find the area of the triangle by substituting in the expression given in Art. 9, but the following is a better method.

Suppose BA to cut the axis of x in D: draw AE, BF perpendicular to the axis of x, and CG perpendicular to AB.

Then the area required is $\frac{1}{2}AB \cdot CG$.

Now $AB = FE \sec ODA = \sqrt{2}$, and GC is $\dfrac{3+3-3}{\sqrt{2}}$ or $3\sqrt{2}$.

Hence the required area is 3.

The figure and geometrical construction are not absolutely essential, but are inserted for the sake of clearness.

3. Find the ratio in which the line joining the points (3, 4), (4, 3) is cut by $y = mx$, and the value of m when it is bisected.

Let the ratio be $p : q$, and let the coordinates of the point of section be x', y'.

Then (Art. 8) $x' = \dfrac{4p+3q}{7}$, $y' = \dfrac{3p+4q}{7}$.

But this point is on $y = mx$

$\therefore\ 3p + 4q = m(4p + 3q) \ \therefore\ \dfrac{p}{q} = \dfrac{3m-4}{3-4m}$.

If the line is bisected $\dfrac{p}{q} = 1$, $\therefore\ m = 1$.

Examples III, *b*.

1. Find the distance of the point (2, 3) from the line
$$x + y = 1.$$

2. Find the distance of the point (3, 0) from the line
$$\frac{x}{2} + \frac{y}{3} = 1.$$

3. Find the distance of the point (0, 1) from the line
$$x - 3y = 1.$$

4. Find the distance of the point (– 1, 3) from the line
$$3x + 4y + 2 = 0.$$

5. Find the distance of the point $(-a, -b)$ from the line

$$\frac{x}{a}+\frac{y}{b}=1.$$

6. Find the distance of the point (a, b) from the line

$$ax-by=0.$$

7. Find the distance of the point (h, k) from the line

$$Ax+By+C=D.$$

8. Find the distance of the origin from the line

$$hx+ky=c^2.$$

9. Find the distance of the point (h, k) from the line

$$hx+ky=c^2.$$

10. Find the distance of the point $(a, 0)$ from the line

$$y=mx+\frac{a}{m}.$$

11. Find the distance of the point h, k, from the line

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1.$$

12. Find the coordinates of the foot of the perpendicular in each of the preceding cases.

13. Find the coordinates of the points of intersection of the four lines in questions 1—4.

14. Find the distance between the following pairs of straight lines

$$y=(x-a)\tan\alpha, \text{ and } y=(x-b)\tan\alpha;$$

$$\frac{x}{a}+\frac{y}{b}=2 \text{ and } \frac{x}{a}+\frac{y}{b}+2=0;$$

$$5x-4y=3 \text{ and } 4y-5x=4;$$

$$x+y+2=0 \text{ and } x+y=1;$$

$$3x+4y=5 \text{ and } 3x+4y+1=0.$$

15. If θ be the angle between the lines whose equations are $lx+my=d$, $l'x+m'y=d'$ respectively, then

$$\cos\theta=\frac{ll'+mm'}{(l^2+m^2)^{\frac{1}{2}}(l'^2+m'^2)^{\frac{1}{2}}}.$$

16. Find the area of the triangle contained by the lines

$$x = y, \; x + y = 0, \; x = c.$$

17. Find that contained by the lines

$$x + y = a, \; 2x = y + a, \; 2y = x + a.$$

18. Find that contained by the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \; y = 2x + b, \; x = 2y + a.$$

19. Find that contained by the lines

$$y = mx + b, \; x = my + a, \; \frac{x}{a} + \frac{y}{b} = 1.$$

20. Find that contained by the lines

$$4x - 3y = 1, \; 12x + 5y = 17, \; 12x - 5y = 3.$$

21. Find the equations to the straight lines which pass through the point (h, k), and form with the axes a triangle of given area.

22. Find the equations to the straight lines which pass through the intersection of $4x + 3y = 7$, and $3x - 4y + 1 = 0$ and are parallel and perpendicular to $4x - 3y = 0$.

23. If the straight lines $4x + 3y = 7$, $3x - 4y + 1 = 0$, cut the axes of x and y in A, B, C, D respectively and intersect in E, find the point within the angle AEC which is at a distance 1 from each.

24. Find all the points which are at a distance d from each of the two straight lines $y = mx + c$, $x \cos \alpha + y \sin \alpha = p$.

25. Find the point the distances of which from $y = mx + c$, $m(y - c) + x = 0$ are a and b respectively, and which lies in that angle formed by those lines which is opposite to that which contains the origin.

26. Find the coordinates of the point in which the line joining the points (3, 4), (4, 5) is cut by the line $\frac{x}{a} + \frac{y}{b} = 1$, and the relation between a and b when it is bisected at that point.

27. Find the coordinates of the point in which the line joining $(1, 3)$, $(3, 1)$ is cut by the line joining $(-a, 0)$, $(0, b)$, and find a and b when it is bisected there, and the line joining the point to $(-a, 0)$ is bisected by the axis of y.

28. Find the ratio in which the line joining the points $(1, 2)$ $(2, 1)$ is cut by that joining $(3, 3)$ and $(1, 1)$.

29. Find the equations to the sides of the equilateral triangles of which $(2, 2)$ $(5, 1)$ are vertices, and the other vertex is on the side remote from the origin.

30. Find the equations to the sides and diagonals of the parallelogram, 3 of whose angular points taken in order are $(a, 0)$, $(2a, 2b)$, $(0, b)$.

CHAPTER IV.

GROUPS OF STRAIGHT LINES.

32. To find the equation to a straight line which passes through the intersection of two given straight lines.

Let the equations to the two given lines be

$$lx + my = d \text{ .... (1)},$$

$$l'x + m'y = d' \text{ .... (2)}.$$

Then $$lx + my - d = k(l'x + m'y - d') \text{ .... (3)}$$

is the equation required, where k may have any value.

For this is the equation to *some* straight line, since it is of the first degree, and it is true when (1) and (2) are satisfied simultaneously, it therefore passes through the only point at which they are both true, the point of intersection of the two straight lines.

Since there are an infinite number of straight lines which pass through a given point, k is indeterminate, and is fixed in any case by the circumstances of the problem.

For instance, we will find the equations to the two straight lines which bisect the angles between (1) and (2).

Let (xy) be a point on one of these lines, p_1, p_2 the perpendiculars on the lines from (xy), then $p_1 = \pm p_2$.

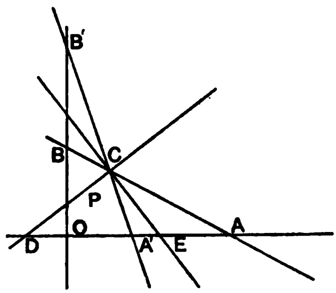
But, (Art. 30)

$$p_1 = \frac{d - lx - my}{(l^2 + m^2)^{\frac{1}{2}}}, \quad p_2 = \frac{d' - l'x - m'y}{(l'^2 + m'^2)^{\frac{1}{2}}}:$$

Hence the equations are

$$\frac{lx + my - d}{(l^2 + m^2)^{\frac{1}{2}}} = \pm \frac{l'x + m'y - d'}{(l'^2 + m'^2)^{\frac{1}{2}}}.$$

To distinguish between the two lines we will discuss them by means of a figure.



Let AB, $A'B'$ be the two straight lines intersecting in C.

Draw DC, EC bisecting the angles between them.

Let P be a point (xy) on either DC or EC.

It is clear that the two straight lines divide the plane into four compartments, and that the origin lies in one of these, unless either d or d' is zero, when it is on one of the lines.

We have also seen that when a point and the origin lie on the same side of a straight line, the perpendicular from the point is positive.

Suppose the origin to lie in the same compartment as D. Then from every point in the straight line DC, the perpendiculars have the same sign, from every point in EC, opposite signs.

The equation $\dfrac{lx + my - d}{(l^2 + m^2)^{\frac{1}{2}}} = \dfrac{l'x + m'y - d'}{(l'^2 + m'^2)^{\frac{1}{2}}}$ represents therefore that bisector which passes through the compartment in which the origin lies.

Exactly in the same way we may prove that the equation

$$lx + my - d = k\,(l'x + m'y - d')$$

represents a straight line through the intersection of the two given straight lines which is in the same compartment as the origin if k is positive, and in the adjacent compartments if k is negative. If d and d' are not

positive, they can be made so by changing the sign of every term.

The figure is drawn to represent the lines $x + 2y = 3$, $3x + y = 4$: the student should investigate the equations to DC and EC for these, and determine where they cut the axes.

k is here called an *arbitrary* constant: it is constant for the same lines, but different values may be assigned to it making the line vary.

The equation may then be considered to represent a group of straight lines always passing through a fixed point.

33. Conversely, if the equation to a straight line contains an arbitrary constant in the first degree, the straight line always passes through a fixed point

The equation must be of the form

$$Ax + By + C + k(A'x + B'y + C') = 0,$$

where k is the arbitrary constant.

Now if $Ax + By + C = 0$, $A'x + B'y + C' = 0$ simultaneously, the equation is true for all values of k, that is to say, the straight line passes through the intersection of the straight lines denoted by these equations.

EXAMPLES WORKED OUT.

1. Find the equation to the line which passes through the intersection of $ax + by + c = 0$, $a'x + b'y + c' = 0$, and is parallel to $lx + my = d$.

The equation to any straight line through the intersection of the first pair is

$$ax + by + c = k(a'x + b'y + c') \text{ where } k \text{ is variable.}$$

Writing this in the form

$$(a - ka')x + (b - kb')y + c - kc' = 0,$$

we see that if it is parallel to $lx + my = d$,

$$(a - ka')\,m = (b - kb')\,l, \text{ or } k = \frac{am - bl}{a'm - b'l}.$$

The required equation is therefore

$$(a'm - b'l)\,(ax + by + c) = (am - bl)\,(a'x + b'y + c')$$

or
$$lx + my + \frac{l\,(bc' - b'c) + m\,(a'c - ac')}{a'b - ab'} = 0.$$

2. Find the equation to the diagonals of the parallelogram whose sides have for their equations

$$2x + y = 9 \ldots\ldots (1), \qquad 2x + y + 3 = 0 \ldots\ldots (3),$$
$$5y - 2x = 9 \ldots\ldots (2), \qquad 5y - 2x + 3 = 0 \ldots\ldots (4).$$

The equation to any straight line through the intersection of (1) and (2) is $2x + y - 9 + k\,(5y - 2x - 9) = 0$.

Similarly, the equation to any straight line through the intersection of (3) and (4) is $2x + y + 3 + k'\,(5y - 2x + 3) = 0$.

Equate coefficients; then $\dfrac{2\,(1 - k)}{2\,(1 - k')} = \dfrac{1 + 5k}{1 + 5k'} = -\dfrac{9\,(1 + k)}{3\,(1 + k')}$.

Whence $k = k' = -1$ and the diagonal is $x = y$.

Similarly the equation to the other diagonal may be written in either of the forms

$$2x + y - 9 + m\,(5y - 2x + 3) = 0,$$
$$2x + y + 3 + m'\,(5y - 2x - 9) = 0.$$

If these are identical $\dfrac{1 - m}{1 - m'} = \dfrac{1 + 5m}{1 + 5m'} = \dfrac{3m - 9}{4 - 9m'}$,

whence $m = m' = 1$ and the required equation is $y = 1$.

Examples IV, *a*.

1. Find the equations to the bisectors of the angles between the following pairs of lines, reducing them in each case to a simple form:

(i) $x + y = a$ and $y = x$. (ii) $y = mx + c$, and $my + x = c$.

(iii) $3x + 4y = 5$ and $4x + 3y = 5$.

(iv) $x = a$ and $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(v) $x\cos\alpha + y\sin\alpha = p$ and $x\cos\beta + y\sin\beta = p$.

(vi) $x + y = 2$, $x + 3y = 4$. (vii) $\frac{x}{a} + \frac{y}{b} = 1$ $\frac{x}{b} - \frac{y}{a} = 1$.

(viii) $y = x + 1$, and $y = 2x + 2$.

2. Find the equation to the straight line which passes through the intersection of $3x + 4y = 7$, $4x - 8y = 2$ and through the origin.

3. Find the equation to the straight line which passes through the intersection of $ax + by + c = 0$, $a'x + b'y + c' = 0$, and through the origin.

4. Find the equation to the straight line which passes through the point $(2, 2)$ and the intersection of the lines $3x - 4y + 1 = 0$, and $5x - 4y - 1 = 0$.

5. Find the equation to the straight line which passes through the point $(2, 1)$ and the intersection of the lines $3x + 4y = 11$, and $x - 3y + 5 = 0$.

6. Find the equation to the straight line which passes through the point (h, k), and through the intersection of the lines $\frac{x}{h} + \frac{y}{k} = 1$ and $y = mx + k$.

7. Find the equation to the straight line which passes through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $y = mx$ and is parallel to the axis of x.

8. Find the equation to the straight line which passes through the intersection of $3y + 7x = 5$, and $4x - y = 10$, and is parallel to the axis of y.

9. Find the equation to the straight line which passes through the intersection of $x - 3y = 4$, and $3x + y = 7$, and is parallel to $7x - y = 4$.

10. Find the equation to the straight line which passes through the intersection of $l_1x + m_1y = d_1$, $l_2x + m_2y = d_2$ and is parallel to $l_3x + m_3y = d_3$.

11. Find the equation to the straight line which passes through the intersection of $3x - 2y = 4$ and $7x - 3y = 1$, and is perpendicular to the former.

12. Find the equation to the straight line which passes through the intersection of $4x + y = 6$, $x - 3y = 2$, and is perpendicular to the latter.

13. Find the equation to the straight line which passes through the intersection of $x = 4$ and $x + y = 7$, and is perpendicular to $y = 4x + 3$.

14. Find the equation to the straight line which passes through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $y = mx$ and is perpendicular to the former.

15. Find the equation to the straight line which passes through the intersection of $x + y = 3$, $x - y = 1$, and also through that of $3x + 2y = 12$, and $7x - 3y + 1 = 0$.

16. Find the equation to the straight line which passes through the intersection of

$$x \cos \alpha + y \sin \alpha = p, \ x \cos \beta + y \sin \beta = p,$$

and also through that of $y = mx + c$, $x = my + c$.

17. Find the fixed points, independent of k, through which the following straight lines pass:

(i) $lx + my = k(l'x + m'y)$.

(ii) $3x + 4y - 7 + k(x - 2y + 1) = 0$.

(iii) $x - y + k(x + 4y - 10) = 0$.

(iv) $\frac{x}{a} - \frac{y}{b} + k\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$.

(v) $y - mx + c = k\{3x + m(y + c)\}$.

(vi) $7x + 17y + 3 + k(x + 2y) = 0$.

18. Shew that the equations $x - y - 1 + k(3x - 7y + 5) = 0$, and $7x = 3y + 15 + p(x + y - 5)$ represent the same group of straight lines.

19. Find the equations to the diagonals of the parallelogram whose sides have for their equations $y = 3$, $y + 2 = 0$, $x + y = 5$, $x + y = 1$.

20. Find the equations to the diagonals of the parallelogram when the equations to the sides are

$$x + 3y = 5, \ x + 3y + 5 = 0, \ 3y - x = 2, \ 3y - x + 2 = 0.$$

21. Find the equations to the diagonals, when those to the sides are

$$x=a,\ x=b,\ x\cos\alpha+y\sin\alpha=p_1,\ x\cos\alpha+y\sin\alpha=p_2.$$

22. Three consecutive angular points of a parallelogram are $(a, 0)$ (h, k), $(0, b)$: find the other angular point, and the equations to the diagonals.

34. *To find the condition that three points may be on the same straight line.*

Let (x_1y_1), (x_2y_2), (x_3y_3), be the coordinates of the points; then, since these points lie on a straight line, the area of the triangle of which they are the angular points must be equal to 0, but twice the area of this triangle is equal to

$$x_1y_2-x_2y_1-x_2y_3-x_3y_2+x_3y_1-x_1y_3;\quad \text{(Art. 9.)}$$

$$\therefore\ x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3=0$$

is the condition required.

Generally however the best way of proving that three points lie on a straight line, is to write down the equation to the line through two of the points and prove that the coordinates of the third satisfy it.

35. *To find the condition that three straight lines pass through the same point.*

This is the same as finding the conditions that the same values of x and y shall satisfy the equations to the three lines.

Let these three equations be

$$l_1x+m_1y=d_1\ldots\ldots\ldots\ldots\ldots\ldots\text{(1)},$$

$$l_2x+m_2y=d_2\ldots\ldots\ldots\ldots\ldots\ldots\text{(2)},$$

$$l_3x+m_3y=d_3\ldots\ldots\ldots\ldots\ldots\ldots\text{(3)}.$$

Multiply (2) by λ, (3) by μ, and add;

$$\therefore\ (l_1+\lambda l_2+\mu l_3)\,x+(m_1+\lambda m_2+\mu m_3)\,y=d_1+\lambda d_2+\mu d_3$$

for all values of λ and μ, and $\therefore$ when $l_1+\lambda l_2+\mu l_3=0$, and $m_1+\lambda m_2+\mu m_3=0$.

But then $d_1 + \lambda d_2 + \mu d_3 = 0$ also.

Now if $l_1 + \lambda l_2 + \mu l_3 = 0$, and $m_1 + \lambda m_2 + \mu m_3 = 0$,

$$\lambda = \frac{l_3 m_1 - l_1 m_3}{l_2 m_3 - l_3 m_2},$$

$$\mu = \frac{l_1 m_2 - l_2 m_1}{l_2 m_3 - l_3 m_2};$$

$$\therefore \; d_1 (l_2 m_3 - l_3 m_2) + d_2 (l_3 m_1 - l_1 m_3) + d_3 (l_1 m_2 - l_2 m_1) = 0$$

is the condition required.

Generally, however, the best way is to find the point of intersection of two straight lines, and to make its coordinates satisfy the third.

Such straight lines are said to be concurrent.

Problems which prove that three straight lines pass through a point may generally be much simplified by a judicious choice of axes.

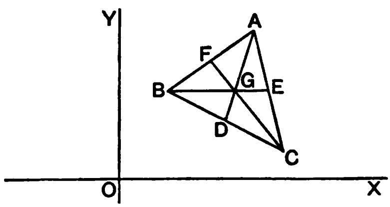
36. *Examples of concurrent lines.*

We will exemplify the methods of some of the preceding articles by proving some of the well-known properties of a triangle, namely that the three sets of three straight lines passing through the angles, and (i) bisecting the opposite sides, (ii) perpendicular to the sides, (iii) bisecting the angles, are concurrent.

i. Let ABC be a triangle, D, E, F the middle points of the sides, and let the coordinates of A, B, C be $(h_1 k_1)$, $(h_2 k_2)$, $(h_3 k_3)$ respectively, any axes being taken, then those of D, E, F will be

$$\frac{h_2 + k_3}{2} + \frac{k_2 + k_3}{2}, \text{ \&c.}$$

(Art. 8);



therefore the equation to AD is (Art. 22)

$$y\left(h_1 - \frac{h_2 + h_3}{2}\right) - x\left(k_1 - \frac{k_2 + k_3}{2}\right) = h_1 \frac{k_2 + k_3}{2} - k_1 \frac{h_2 + h_3}{2},$$

or

$$y\{2h_1-(h_2+h_3)\}-x\{2k_1-(k_2+k_3)\}=h_1(k_2+k_3)-k_1(h_2+h_3) \qquad \text{.........(1)}.$$

Now the equation to BE may be obtained from this by a *cyclic change of suffixes:* that is if we write 2 for 1, 3 for 2, 1 for 3, we pass from A to B, B to C, C to A, and from D to E, E to F, and F to D.

Hence the equations to BE, and CF may be written down. They must be

$$y\{2h_2-(h_3+h_1)\}-x(2k_2-(k_3+k_1)\}=h_2(k_3+k_1)-k_2(h_3+h_1) \qquad \text{.........(2)},$$

$$y\{2h_3-(h_1+h_2)\}-x\{2k_3-(k_1+k_2)\}=h_3(k_1+k_2)-k_3(h_1+h_2) \qquad \text{.........(3)}.$$

Now if we add these three equations together, both sides become identically zero.

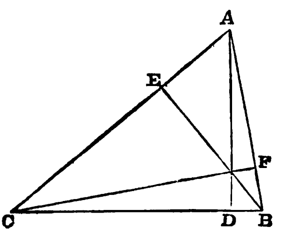
This shews that the three lines represented *pass through a point,* for if we add the two first together, we get an equation which represents a straight line through the intersection of that pair, but by so adding them we get the equation to the third, which therefore passes through that point of intersection.

We might have proved this by finding the point of intersection of two: it is easily seen to be

$$\frac{h_1+h_2+h_3}{3},\ \frac{k_1+k_2+k_3}{3}.$$

This point is called the centroid, or centre of gravity, of the triangle, and the intersecting lines are called medians.

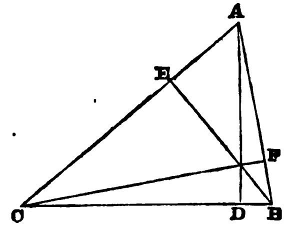
ii. Let AD, BE, CF be perpendicular to the sides. Take CB as axis of x, AD as that of y; let $DB=a$, $DC=a'$, $DA=b$.



Then the equations to AB, AC are

$\frac{x}{a}+\frac{y}{b}=1$ and $\frac{y}{b}-\frac{x}{a'}=1$ respectively.

Hence the equations to CF, BE which are perpendicular to these through $(-a', 0)$ $(a, 0)$ respectively are (Art. 28)



$$a(x+a') - by = 0,$$

$$a'(x-a) + by = 0.$$

The line whose equation is $(a+a')x = 0$, or $x = 0$, which is obtained by adding these equations must pass through the intersection of CF and BE. (Art. 32.)

But $x = 0$ is the equation to AD, which proves the theorem. The point where these three straight lines intersect is called the orthocentre.

iii. Next, to prove that the bisectors of the angles are concurrent. We will use the property that if a straight line bisect an angle, the perpendiculars from any point on it on the arms of the angle must be equal.

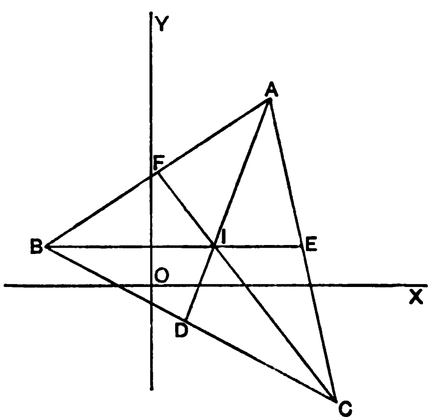
Take any rectangular axes, the origin being within the triangle, and let the equations to BC, CA, AB be

$$x\cos\alpha + y\sin\alpha = p \quad \ldots\ldots(1),$$

$$x\cos\beta + y\sin\beta = q \quad \ldots\ldots(2),$$

$$x\cos\gamma + y\sin\gamma = r \quad \ldots\ldots(3) \text{ respectively.}$$

Let AD, BE, CF be the bisectors.



Then the equation to AD is

$$x \cos \beta + y \sin \beta - q = x \cos \gamma + y \sin \gamma - r,$$

since these expressions are the lengths of the perpendiculars from (x, y) on (2), (3) respectively, changed in sign.

Similarly the equation to BE is

$$x \cos \alpha + y \sin \alpha - p = x \cos \gamma + y \sin \gamma - r;$$

therefore at the intersection of these two lines

$$x \cos \alpha + y \sin \alpha - p = x \cos \beta + y \sin \beta - q.$$

But this is the equation to CF, which therefore passes through the intersection of AD and BE.

It may be proved in the same way that the bisectors of any pair of exterior angles of a triangle intersect on the bisector of the third angle.

In applying theorem (3) to any triangle whose sides are given, the student should be careful in drawing a figure to represent the sides.

If these sides are only given generally by means of such equations as $lx + my = d$, it is impossible to tell in which compartment the origin is situated, and therefore to distinguish between the external and internal bisectors, but if we draw a figure, and notice the compartments in which O is situated we can make the necessary distinction in any given case.

The point I where the internal bisectors intersect, is the centre of the inscribed triangle, or the incentre. The other points are called excentres.

Examples IV, *b*.

1. Write down the coordinates of the centroid when the coordinates of the angular points are (i) $(0, 0)$, $(a, 0)$, $(0, b)$.
(ii) $(3, 4)$, $(1, 2)$, $(-4, -6)$. (iii) (a, b), $(-a-0)$ $(a-b)$.

2. Find the coordinates of the centroid in the triangles, the equations to whose sides are as follows:
(i) $x=0,\ y=0,\ x+y=3$. (ii) $y=3,\ x+y=0,\ x-y=0$.
(iii) $x=0,\ \frac{x}{6}+\frac{y}{2}=1,\ x-y-2=0$.
(iv) $\frac{x}{a}+\frac{y}{b}=1,\ y=mx,\ my+x=a$.

3. Prove that the straight lines at right angles to the sides of any triangle from their middle points are concurrent, and that their point of intersection, (the circumcentre), is equally distant from the angular points. Take the angular points as $(a, 0)$, $(-a, 0)$ (h, k).

4. Find the orthocentre and centroid in the triangle in question 3, and hence shew that in any triangle, if O be the circumcentre, G the centroid, P the orthocentre, OGP is a straight line such that $2OG=GP$.

5. Find the orthocentre and circumcentre in the triangles given in question 2.

6. Find the orthocentre in each of the following triangles:

(i) $x=0,\ y=mx,\ \frac{x}{a}+\frac{y}{b}=1$.

(ii) $x+y=1,\ y=2x,\ x=2y$.

(iii) $x=a,\ \frac{x}{a}+\frac{y}{b}=1,\ y=mx+b$.

(iv) $x\cos a+y\sin a=p,\ x+b=0,\ y=p-x\cos a$.

7. Find the centroid, orthocentre, and centre of inscribed circle, in the triangle whose angular points are $(2, 3)$, $(1, -2)$, $(-3, 1)$.

8. Find the equations to the straight lines which bisect the interior angles of the following triangles, find also the incentres.

(i) $x\cos\frac{\pi}{3}+y\sin\frac{\pi}{3}=1,\ \ x\cos\frac{\pi}{6}+y\sin\frac{\pi}{6}=1,\ \ x\cos\frac{\pi}{3}-y\sin\frac{\pi}{3}=1.$

(ii) $x+y=2,\ x=0,\ y=0.$

(iii) $2y+x=2,\ 2x+y=4,\ y=2x.$

(iv) $x+3y=3,\ 3y-x=3,\ 3x=y.$

(v) $3x+4y=24,\ 3x-4y=12,\ 4x=3y.$

(vi) $x+p=0,\ x\cos\alpha+y\sin\alpha=p,\ x\cos\alpha-y\sin\alpha=p.$

9. The sides of a triangle are $x+y=3$, $y=0$, $y=2x$, find the equations to the lines which bisect the exterior angles.

37. Although the equation to a straight line is of one dimension only, yet it does not follow that every equation of more than one dimension does not represent straight lines.

Take for instance the equation

$$xy-a(x+y)+a^2=0.$$

This may be put in the form $(x-a)(y-a)=0$,

$$\therefore \text{ either } x-a=0, \text{ or } y-a=0,$$

that is to say, the locus of the equation is two straight lines. Generally, whenever an equation can be resolved into simple factors, it is satisfied by putting each factor separately equal to zero, and is therefore the locus of the various lines whose equations are so obtained.

The locus of an equation may be a point, or may be impossible. For instance, let

$$(x-y)^2+(x+y+a)^2=0.$$

Here $x-y=0$, and also $x+y+a=0$, since if the sum of two squares be zero, each of them must vanish;

$$\therefore x=y=-\frac{a}{2}$$

is the only point which satisfies the equation.

Similarly, if we have the equation

$$x^2 + (x - y)^2 + a^2 = 0,$$

the locus is impossible, since no real values of x and y can be found such that $x^2 + (x - y)^2$ shall be negative.

The easiest way of testing an equation of the second degree which does not split up into factors by inspection is to treat it as a quadratic in either x or y.

The general equation of the second degree is generally written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0:$$

we will discuss the conditions that it may represent two straight lines or a point.

38. A homogeneous equation of the second degree always represents two straight lines through the origin, or the origin itself.

For the equation $ax^2 + 2hxy + by^2 = 0$ is equivalent to $ax + hy = \pm (h^2 - ab)^{\frac{1}{2}} y$.

Now if $h^2 - ab$ be positive, this represents two real straight lines through the origin, but if negative, the equation is impossible unless both x and y vanish: it is therefore only true at the origin: in this case it may be said to represent two imaginary lines.

If $h^2 - ab = 0$ the left-hand side becomes a perfect square, so that the equation represents a straight line through the origin, or rather two coincident straight lines.

39. We will discuss some of the properties of these straight lines without actually determining them.

Since any straight line through the origin may be represented by $y = mx$, where m is the tangent of the angle made with the axis of x, we may suppose that the lines are represented by $(y - mx)(y - m'x) = 0$.

Hence, dividing every term in $ax^2 + 2hxy + by^2 = 0$ by b, and equating coefficients, we have

$$m + m' = -\frac{2h}{b}, \quad mm' = \frac{a}{b}.$$

These straight lines are therefore at right angles if $a + b = 0$, for then $mm' = -1$. Hence the equation

$$x^2 - y^2 = \lambda xy$$

always represents a pair of straight lines through the *origin* which are at right angles to each other, and may represent any pair.

Again the angle between the lines $y - mx = 0$, $y - m'x = 0$ is $\tan^{-1}\dfrac{m - m'}{1 + mm'}$ (Art. 28).

Now $\quad (m - m')^2 = (m + m')^2 - 4mm' = \dfrac{4(h^2 - ab)}{b^2}$.

$$\therefore \frac{m - m'}{1 + mm'} = \frac{2(h^2 - ab)^{\frac{1}{2}}}{a + b}.$$

The equation to the pair of straight lines through the origin perpendicular to $ax^2 + 2hxy + by = 0$ must be

$$(my + x)(m'y + x) = 0,$$

or $$mm'y^2 + (m + m')xy + x^2 = 0,$$

that is, multiplying every term by b,

$$ay^2 - 2hxy + bx^2 = 0.$$

40. To find the equation to the straight lines which bisect the supplementary angles between those whose equation is $ax^2 + 2hxy + by^2 = 0$.

Suppose (xy) to be a point on one of these bisectors then the perpendiculars from (xy) on these lines must be equal without regard to sign.

Hence we must have

$$\frac{(y - mx)^2}{1 + m^2} = \frac{(y - m'x)^2}{1 + m'^2} \quad \text{(Art. 32.)}$$

Multiplying up, and rearranging we get

$$y^2(m'^2-m^2)-2xy(m-m')(1+mm')+(m^2-m'^2)x^2=0.$$

Dividing by $m-m'$ and substituting for $m+m'$ and mm', we obtain

$$h(x^2-y^2)-(a-b)xy=0,$$

the equation required.

41. The general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

does not generally represent two straight lines: we will investigate the condition that it shall do so.

Multiply by a, then the equation may be written in the form

$$\begin{aligned} a^2x^2+2a(hy+g)x+(hy+g)^2 &= (hy+g)^2-(aby^2+2afy+ac) \\ &= (h^2-ab)y^2+2(gh-af)y+g^2-ac. \end{aligned}$$

Hence

$$ax+hy+g=\pm\{(h^2-ab)y^2+2(gh-af)y+g^2-ac\}^{\frac{1}{2}}.$$

If the expression on the right-hand side under the fractional index is a perfect square, or a perfect square with its sign changed, the equation will represent two real straight lines or two imaginary lines which intersect in a real point.

The condition for either is

$$(gh-af)^2=(h^2-ab)(g^2-ac),$$

that is, $$abc+2fgh-af^2-bg^2-ch^2=0.$$

The expression on the left is called the discriminant of the general equation of the second degree and is denoted by Δ.

If this condition is fulfilled, the equation represents two real or imaginary straight lines as h^2-ab is positive or negative.

In testing any quadratic equation it is generally much easier to treat it as an ordinary quadratic in x or y than to substitute the coefficients in the discriminant.

The same result might be obtained by finding the condition that the general equation should reduce to the form $(y - mx - d)(y - m'x - d') = 0$, but the process is much more difficult than that given above.

It is easy to see, however, that

$$m + m' = -\frac{2h}{b}, \quad mm' = \frac{a}{b},$$

and therefore, if the general equation of the second degree does represent two straight lines, that the straight lines through the origin parallel to them are represented by

$$ax^2 + 2hxy + by^2 = 0,$$

and therefore that the angle between them, the condition for perpendicularity, and the equation to the lines through the origin which are parallel to the bisectors of the angles are those already investigated.

EXAMPLES. IV, *c*.

1. *ABCD* is a quadrilateral, whose sides, taken in order, are $y = 0$, $2y + 3x = 3$, $2y + x = 2$, $y = x + 1$, if AD, BC meet at E, AB, DC at F, find the equations to AC, BD, and EF.

2. If the sides of the quadrilateral are $y = 0$,

$$y + (x - a)\tan\alpha = 0, \quad y - b + x\tan(\alpha - \beta) = 0, \quad y = x\tan\theta + b:$$

find the equations to the three diagonals.

3. *ABCD* is a rectangle, P any point (hk): straight lines are drawn through A, B, C, D, perpendicular to PA, PB, PC, PD respectively, shew that two of the diagonals of the quadrilateral so formed are parallel to the sides of the rectangle, and that the third is perpendicular to the line joining P to the intersection of the other two.

Take $x = \pm a$, $y = \pm b$, as the equations to the sides.

4. If on the sides of a triangle, taken in turn as diagonals, be constructed parallelograms, the sides of which are parallel to two fixed lines, the other diagonals of these parallelograms will pass through a point. Take the fixed lines $y = mx$, $y + mx = 0$.

5. Interpret the equations (i) $xy = 0$, (ii) $x^2 - y^2 = 0$,
(iii) $x^2y = xy^2$, (iv) $AB(x^2 + y^2) + (A^2 + B^2)xy + Bx + Ay = 0$,
(v) $x^2 \cos^2\theta - y^2 \sin^2\theta = p^2 - 2py \sin\theta$,
(vi) $x^3 + 2x^2y - 2xy^2 - 4y^3 = 0$,
(vii) $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0$,
(viii) $x^2 - 4xy + 5y^2 - 6y + 9 = 0$,
(ix) $x^2 - 4xy + 3y^2 + 6y - 9 = 0$,
(x) $2x^2 - 3xy - 2y^2 + 3x - 3y = 9$.

6. Interpret the equations (i) $(r^2 - a^2)^2 + b^2(\theta^2 - a^2)^2 = 0$,
(ii) $(r - a)^2 + b^2(\theta - \alpha)^2 = 0$,
(iii) $(r - a)^2(\theta - \alpha)^2 + (r - b)^2(\theta - \beta)^2 = 0$.

7. The equation $\dfrac{x^2}{a^2} + \dfrac{b^2}{y^2} + \dfrac{a^2y^2}{b^2x^2} = \dfrac{y^2}{b^2} + \dfrac{a^2}{x^2} + \dfrac{b^2y^2}{a^2y^2}$
represents the sides and diagonals of a parallelogram.

8. The equation $y^2 - 2xy \sec\alpha + x^2 = 0$, represents two straight lines including an angle α.

9. If $ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0$, represents two parallel straight lines, $ab' = a'c$, and $a'b = b'c$.

10. The distance of (x_1y_1) from each of two straight lines through the origin is δ, shew that the straight lines are represented by $(xy_1 - x_1y)^2 = \delta^2(x^2 + y^2)$.

11. If $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$, represent two pairs of straight lines OA, OB and OA', OB', find the conditions (i) that OA may coincide with OA', (ii) OA may be perpendicular to OA', (iii) $AOA' = BOB'$, (iv) $AOB = A'OB'$.

12. Find the condition that $ax^2 + 2hxy + by^2 = 0$, may represent two straight lines such that one makes the same angle with the axis of x that the other does with the axis of y.

13. If $ax^2 + 2hxy + by^2 = c$ represents two straight lines, they must be parallel.

14. If the general equation of the second degree represents two intersecting straight lines, and one of the bisectors of the angles between them pass through the origin

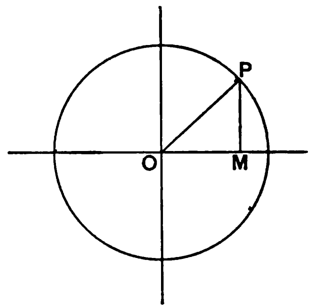
$$h(f^2 - g^2) = fg(b - a).$$

15. If the straight lines represented by the general equation be parallel, then $h^2 = ab$, and $bg^2 = af^2$, and the distance between the lines is $2\left\{\dfrac{g^2 - ac}{a(a+b)}\right\}^{\frac{1}{2}}$.

CHAPTER V.

THE CIRCLE. EQUATIONS.

42. To find the equation to a circle we have to express by means of an equation the condition that the distance of a variable point from a fixed point is constant.



Let the variable point P be (xy), and, first, let the fixed point be the origin, and let the radius be d.

Then, in the figure,

$$OM^2 + PM^2 = OP^2,$$

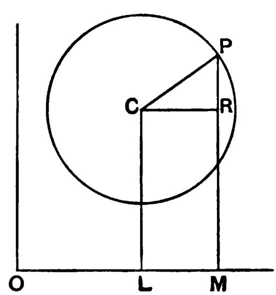
or $$x^2 + y^2 = d^2,$$

the equation required.

This is the form used generally in all questions in which only one circle is involved, and in which we are at liberty to assume any position for the centre.

The student should observe that the point whose coordinates are $d \cos \theta$, $d \sin \theta$ is always on this circle.

43. Let the coordinates of the centre be a, b.



Let P be the point (x, y), C the point (a, b). Draw $CL \perp$ the axis of x, and let CR be drawn parallel to the axis of x to cut the ordinate of P in R.

Then $CR^2 + RP^2 = CP^2$,

but $CR = OM - OL = x - a$,

$$RP = y - b.$$

$\therefore$ the equation is

$$(x-a)^2 + (y-b)^2 = d^2.$$

This is the most general form of the equation to the circle: the student should observe that the straight lines which are denoted by x, y in Art. 42, are now denoted by $x-a$, $y-b$ respectively, and therefore that any result derived from the special form $x^2+y^2=d^2$, can be at once generalised by writing $x-a$, $y-b$, for x and y.

44. The position of the circle for special values of a, b and d should be carefully noted.

We will give a few cases, in each of which a figure should be drawn, and the statements verified.

If $a^2+b^2=d^2$, the origin is on the circle.

If $a^2=d^2$, the circle touches the axis of y, if $b^2=d^2$, the axis of x.

If $a=0$, the centre is on the axis of y.

If $b=0$, the centre is on the axis of x.

45. To find the condition that the general equation of the second degree shall represent a circle.

Let it be the equation to that circle whose centre is the point a, b, and radius d.

Then it must be identical with $(x-a)^2+(y-b)^2=d^2$,

or $$x^2+y^2-2ax-2by+a^2+b^2-d^2=0.$$

There must therefore be no term in xy, and the coefficient of y^2 must be the same as that of x^2.

We may divide by that coefficient, and see that the equation

$$x^2+y^2+2gx+2fy+c=0,$$

is the most general form possible by which a circle can be represented.

g, f, c depend on the position of the centre and the radius.

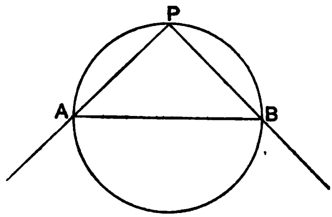
We can write the equation in the form

$$(x+g)^2+(y+f)^2=g^2+f^2-c,$$

which shews that the distance of xy from $-g$, $-f$ is $\sqrt{g^2+f^2-c}$, (Art. 30).

$-g, -f$ are therefore the coordinates of the centre, and the radius is $\sqrt{g^2+f^2-c}$.

46. To find the equation to the circle described on a given straight line as diameter.



Let AB be the given straight line, and P any point on the circle.

Let PA, PB make angles θ, θ' with the axis of x.

Let the coordinates of A, B, P be (x_1, y_1), (x_2, y_2), (x, y) respectively and let $AP = r$, $BP = r'$.

Then (Art. 24) $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$.

Also $\quad x = x_2 + r'\cos\theta'$, $y = y_2 + r'\sin\theta'$.

$$\therefore\ (x-x_1)(x-x_2)+(y-y_1)(y-y_2) = rr'\cos(\theta'-\theta)$$

but $\theta'-\theta = APB$, which is a right angle. (Euc. iii, 31.)

$$\therefore\ (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

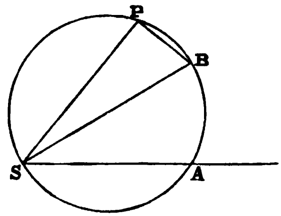
is the equation required.

It is easily seen that this equation does represent a circle whose centre is the mid-point of AB and radius $\frac{AB}{2}$; indeed the above equation might easily have been found from these considerations.

47. *To find the polar equation to a circle.*

(i) Let the centre be pole and the radius d, then the equation is evidently $r = d$.

(ii) Let the pole S be on the circumference, and let the diameter SB make an angle α with the initial line SA.



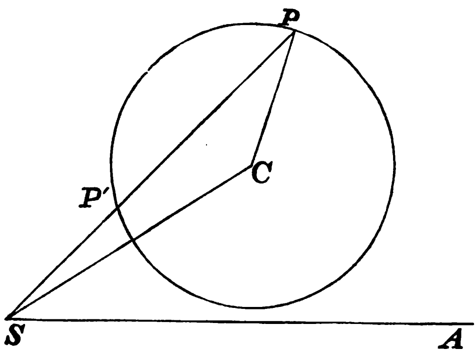
Let the coordinates of P, any point on the circle, be r, θ. Join BP.

Then $\quad SP = SB \cos BSP$,

or $\quad r = 2d \cos(\theta - \alpha)$,

the equation required.

(iii) Let the coordinates of the centre C be l, α.



Then, in the triangle SCP,

$$SP^2 - 2SP \, . \, SC \cos PSC + SC^2 - CP^2 = 0,$$

or

$$r^2 - 2lr \cos(\theta - \alpha) + l^2 - d^2 = 0,$$

the equation required.

N.B. In this equation if r_1, r_2 be the two values, SP, SP' of r corresponding to any value of θ, we know that

$$r_1 r_2 = l^2 - d^2.$$

This proves that, if from any point a straight line be drawn cutting a circle, the rectangle contained by its segments is constant.

If the pole be within the circle, $l^2 < r^2$ and $\therefore$ r_1, r_2 have opposite signs, that is, they are drawn in opposite directions.

EXAMPLES DISCUSSED.

i. To find the equation to the circle which passes through points whose coordinates are given.

There are 3 methods by which we may proceed.

We may assume that the equation to the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

We may then write down 3 equations in $g, f,$ and c by substituting the coordinates of the given points for x and y, and solve these equations.

This method is generally tediously long, unless one of the given points is the origin, in which case $c = 0$.

Or, we may eliminate g, f, c from the equations we have obtained.

This is the best method for the student to adopt, if he is familiar with the methods of elimination between linear equations, in which case he may probably be able at once to write down the result of elimination.

Or again, we may assume that the coordinates of the centre are a and b, and write down the conditions that the distance of (a, b) from the 3 given points is the same.

This will give two equations for determining a and b, and the radius will be the distance between (a, b) and one of the given points.

ii. To find the equation to the chord joining the point

$$(d \cos \alpha, d \sin \alpha), (d \cos \beta, d \sin \beta).$$

Substitute these values for $(x_1 y_1)$, $(x_2 y_2)$ in the equation

$$y (x_2 - x_1) - x (y_2 - y_1) = x_2 y_1 - x_1 y_2.$$

It becomes,

$$y (\cos \beta - \cos \alpha) - x (\sin \beta - \sin \alpha) = d (\cos \beta \sin \alpha - \cos \alpha \sin \beta),$$

or

$$2x \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} + 2y \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2} = d \sin (\alpha - \beta),$$

which reduces to

$$x \cos \frac{\alpha + \beta}{2} + y \sin \frac{\alpha + \beta}{2} = d \cos \frac{\alpha - \beta}{2}.$$

EXAMPLES V.

1. Determine the radii of the circles denoted by the following equations:

$$x^2 + y^2 = 9a^2,$$
$$(x+y)^2 + (x-y)^2 = 8a^2,$$
$$\frac{x^2 + y^2}{a^2 + b^2} = 1.$$

2. Determine the coordinates of the centre, and the radius of each of the circles denoted by the following equations:

$$x^2 + y^2 - 2a(x-y) = c^2,$$
$$x^2 + y^2 + ax + by = a^2 + b^2,$$
$$x^2 + y^2 - 3x - 4y + 4 = 0,$$
$$x^2 + y^2 + 8x + 6y = 0,$$
$$x^2 + y^2 = 6x + 8y + 11,$$
$$x^2 + y^2 = ax,$$
$$x^2 + y^2 = by,$$
$$x^2 + y^2 = a(x+a),$$
$$x^2 + y^2 = ax + by.$$

3. Find the equation to the circle whose radius is a, and coordinates of the centre $a, -a$.

4. To that whose radius is $5a$, and coordinates of centre $3a, 4a$.

5. To that whose radius is c, and coordinates of centre $b+c, b-c$.

6. To that which passes through the origin and cuts off lengths a, b from the axes, on the positive sides.

7. Find the equations to the circles which pass through the origin and the following pairs of points; (3, 4) and (4, 3); (3, 4) and (– 4, 3); (2, 1) and (– 2, 1); (h, k) and $(h, -k)$.

8. Find the equations to the circles which pass through the following sets of three points; (1, 1), (1, – 1), (2, 3); (2, 3) (– 2, 3), (1, 1); (2, 0), (0, – 3), (2, – 3); (2, 0), (0, 3), (– 2, – 3).

9. Find the equations to the circle which passes through the points (3, 1), (1, 3) and whose radius is $\sqrt{2}$.

10. Find the equations to the two circles which pass through the points (4, 0) (−4, 6) and whose radius is 13.

11. Find the equation to the four circles whose radius is $\sqrt{2a}$, and which cut off chords from each axis equal to $2a$.

12. Find the equation to the circle which passes through the origin and cuts off equal lengths a from the lines $y = x$, $x + y = 0$.

13. Find the equation to the circle which circumscribes the triangle whose sides have for their equations, $y = 0$, $y = \mu x + b$, $\frac{x}{a} + \frac{y}{b} = 1$, respectively.

14. If the circle represented by $x^2 + y^2 + 2gx - 6y = 132$ passes through the point (2, 8) find the value of g and the radius.

15. Find the radius, when the circle represented by

$$x^2 + y^2 = 2(x + y) + c$$

passes through the point (2, 3).

16. Find the equation to the circle whose centre is on the line $3x + 4y = 7$, and which passes through the points (0, 4) (4, 2).

17. Find the equation to the circle whose centre is the origin, and which cuts off a chord of length $2c$ from $\frac{x}{a} + \frac{y}{b} = 1$.

18. Find the equations to the circles which pass through the origin, have their centres on the axis of x, and cut off a chord of length 10 from $3x + 4y = 5$.

19. The centre of a circle is on the line $\frac{x}{a} + \frac{y}{b} = 1$, its radius is $\sqrt{a^2 + b^2}$ and it cuts off a chord of length $2a$ from the axis of x, find its equation.

20. Find the equation to the circle whose diameter is the common chord of $x^2 + y^2 - 2x - 2y = 2$, and $x^2 + y^2 + 2x + 2y = 2$, and shew that it passes through the centres of each of these circles.

21. The two circles whose equations are

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 - 2gx - 2fy + c = 0,$$

cannot intersect unless c is negative and then $x^2 + y^2 + c = 0$ is the equation to the circle on the common chord as diameter.

22. The equation $r^2 + 2r(A \cos\theta + B \cos\theta) + C = 0$ always represents a circle, provided $A^2 + B^2 > C$.

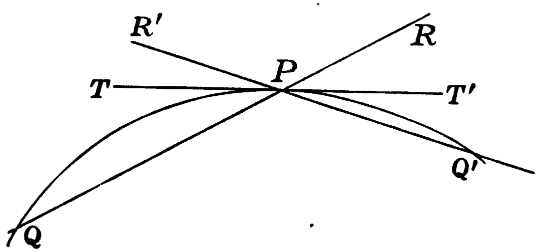
23. ABC is an equilateral triangle, the length of a side being a, find the equation to its circumscribing circle, (i) when the coordinates are rectangular, AB the axis of x and the middle point of AB origin, (ii) when A is pole, AB the initial line.

CHAPTER VI.

CHORDS AND TANGENTS.

48. WE might proceed to find the equation to the tangent to a circle at any point from the consideration that it is perpendicular to the diameter through that point; we will, however, give a definition of a tangent, and a method of finding its equation, which will be applicable to all curves.

DEF. Let QPQ' be a curve, P a point on it, Q any other point on it; draw the secant QPR; let Q move



along the curve to P; then the limiting position of the secant QPR, when Q *moves up to and ultimately coincides* with P, is called the tangent to the curve PQP' at P.

It is evident that on whichever side of P we take Q, for every position of Q there is a definite position of PQ; there must therefore be some position when the point Q is neither on *one* side of P nor on the *other*, but *at* P: this position is called the tangent at P.

49. *To find the equation to the tangent at any point of a circle.*

Let the point be (x_1y_1), and the equation to the circle

$$x^2 + y^2 = a^2 \qquad \text{(1)},$$

and first let us find the secant passing through the points (x_1y_1), (x_2y_2).

The equation to the straight line through these points has been already found, it is (Art. 22)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{(2)}.$$

Now if the points P, Q coincide, $x_2 = x_1$; $y_2 = y_1$, and the fraction $\frac{y_2 - y_1}{x_2 - x_1}$ assumes the indeterminate form $\frac{0}{0}$.

We have not, however, introduced the condition that these points should lie on the circle.

Since $$x_1^2 + y_1^2 = a^2 = x_2^2 + y_2^2,$$

$$\therefore\ y_2^2 - y_1^2 = x_1^2 - x_2^2,$$

$$\therefore\ \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

Substituting in equation (2) we obtain

$$y - y_1 + \frac{x_2 + x_1}{y_2 + y_1}(x - x_1) = 0.$$

Now let Q coincide with P, or $x_2 = x_1$, $y_2 = y_1$, and the equation becomes

$$y - y_1 + \frac{x_1}{y_1}(x - x_1) = 0,$$

or $$yy_1 + xx_1 = y_1^2 + x_1^2 = a^2:$$

$$xx_1 + yy_1 = a^2$$

is therefore the equation required.

50. *To find the points where the tangent cuts the axes.*

Putting y and x successively equal to zero in the equation to the tangent, we obtain

$$x = \frac{a^2}{x_1},\ y = \frac{a^2}{y_1}.$$

51. Def. The normal to a curve at any point is the perpendicular to the tangent at that point.

To find the equation to the normal at the point (x_1y_1).

Since the normal is perpendicular to the tangent

$$xx_1 + yy_1 = d^2,$$

its equation must be

$$(x - x_1)\,y_1 - (y - y_1)\,x_1 = 0; \text{ (Art. 26)}$$

or

$$xy_1 - yx_1 = 0.$$

The normal therefore passes through the centre.

52. *To find the condition that the line*

$$lx + my = \delta$$

should touch the circle $x^2 + y^2 = d^2$.

The simplest method of finding the condition that a straight line should touch a circle is to make the perpendicular from the centre equal to the radius.

Now the perpendicular from the origin on $lx + my = \delta$ is

$$\frac{\delta}{\sqrt{l^2 + m^2}}; \quad \therefore \quad \frac{\delta}{\sqrt{l^2 + m^2}} = d \text{ or } \delta^2 = (l^2 + m^2)\,d^2$$

is the condition required.

The straight line $x \cos\theta + y \sin\theta = d$, of course touches the circle, since the perpendicular from the origin is equal to d, and it is the simplest form of the equation to take in all questions in which the point of contact is not involved.

The student should always consider carefully before beginning any problem whether the coordinates of the point of contact are (or are not) involved.

53. We will give another method of finding this condition which is applicable to any curve, and agrees with our definition of a tangent.

If we eliminate y between

$$lx + my = \delta,$$
$$x^2 + y^2 = d^2,$$

we shall obtain a quadratic in x, the roots of which will give us the abscissæ of the points where the line *cuts* the circle.

If the line *touch* the circle, the points of section must coincide, and the roots become equal.

The equation $lx + my = \delta$ may be written

$$my = \delta - lx.$$

Multiply both sides of the equation $x^2 + y^2 = d^2$ by m^2 and substitute for y;

$$\therefore\ m^2x^2 + (\delta - lx)^2 = m^2d^2,$$

or
$$(l^2 + m^2)\,x^2 - 2\delta lx + \delta^2 - m^2d^2 = 0.$$

If this equation has equal roots

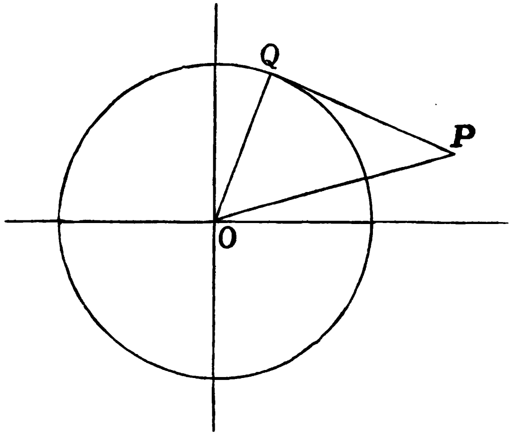
$$l^2\delta^2 = (l^2 + m^2)\,(\delta^2 - m^2d^2),$$

or
$$\delta^2 = (l^2 + m^2)\,d^2,$$

the condition we obtained before.

54. *To find the length of the tangent drawn from the point xy to the circle*

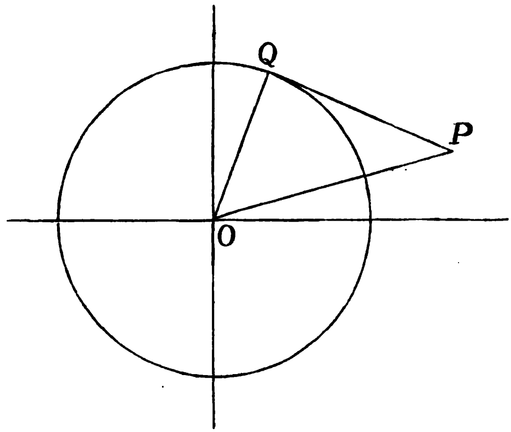
$$x^2 + y^2 = d^2.$$



Let PQ be the tangent, then

$$PQ^2 = OP^2 - OQ^2 = x^2 + y^2 - d^2.$$

Hence that expression which is equal to zero when the point (xy) is on the circle, is equal to the square of the length of the tangent from (xy) when the point is without the circle.



If the point be within the circle,

$$x^2 + y^2 - a^2 \text{ is negative,}$$

its square root is therefore impossible; this shews that no geometrical tangent can be drawn to the circle from a point within it.

55. We saw (Art. 24) that if a straight line make an angle θ with the axis of x and pass through a point (x_1y_1), we may write $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$, where r is the distance between the points (xy) and (x_1y_1).

Now write these values in the equation to the circle and arrange by powers of r; it becomes

$$r^2 + 2(x_1\cos\theta + y_1\sin\theta)r + x_1^2 + y_1^2 - a^2 = 0.$$

Hence if r_1, r_2 be the roots of this equation

$$r_1r_2 = x^2 + y^2 - a^2$$

for all values of θ: this proves Euc. III. 35—37.

EXAMPLES VI, *a*.

1. Write down the equations to the tangents to the circle

$$x^2 + y^2 = d^2$$

which pass through the points on the circle,

$$a,\ 0;\ \frac{3}{5}d,\ -\frac{4}{5}d;\ -h,\ k.$$

2. Find the equations to the tangents to the circle

$$x^2 + y^2 = d^2$$

which have the following properties respectively;

(i) Make a given angle with the axis of x:

(ii) Are parallel to $\frac{x}{h} + \frac{y}{k} = 1$:

(iii) Are perpendicular to

$$Ax + By + C = 0:$$

(iv) Pass through a given point on the axis of y:

(v) Are at a distance δ from the point $c \cos \alpha$, $c \sin \alpha$:

(vi) Cut off a triangle of area a^2 from the axes.

3. Find the condition that the lines

$$Ax + By + C = 0,$$
$$y - y_1 = (x - x_1) \tan \alpha,$$
$$y = mx + c,$$

should touch the circle $x^2 + y^2 = d^2$.

4. Find the values of m when the straight line

$$y = mx + 4$$

touches the circle $x^2 + y^2 = 4$.

Find also the points of contact.

5. Find the relation between l and m, when $lx + my = 1$ touches the circle $x^2 + y^2 = d^2$.

6. If $lx + my = d$ touch $x^2 + y^2 = d^2$, then $l^2 + m^2 = 1$.

7. If $lx+3y=25$ touch the circle $x^2+y^2=25$, find the value of l.

8. Find the value of m when $my=x-10$ touches the circle $x^2+y^2=25$.

9. Find the value of a when $12x+5y=13$ touches the circle $x^2+y^2=a^2$.

10. Find the points in which the straight line $x+y=7$ cuts the circle $x^2+y^2=25$.

11. Write down the equations to the tangents at the points found in the preceding question, and shew that the straight line joining their intersection to the centre bisects the chord joining them orthogonally.

12. Shew that the equation to the chord joining the points (x_1y_1), (x_2y_2) on the circle $x^2+y^2=d^2$, may be written in the form $x(x_1+x_2)+y(y_1+y_2)=x_1x_2+y_1y_2+d^2$.

13. Find the point of intersection of the tangents at the points (x_1y_1) (x_2y_2), and shew that the straight line joining this point to the centre bisects the chord at right angles.

14. If T be the point of intersection of tangents to a circle at P and Q, whose centre is C, and if CT cut PQ in M, prove analytically that $CM \,.\, CT=CP^2$.

56. *To find the equation to the tangent to the circle whose equation is*

$$x^2+y^2+2gx+2fy+c=0.$$

(i) This equation may be made to depend on those already found by writing the equation to the circle in the form

$$(x+g)^2+(y+f)^2=g^2+f^2-c.$$

Comparing this equation with $x^2+y^2=d^2$, we see that the coordinates of the centre are $(-g,-f)$ and the radius $\sqrt{g^2+f^2-c}$.

Hence $x+g$, $y+f$ in the first equation have the same geometrical meaning as x, y in the second, namely the intercepts cut off from two fixed lines at right angles through the centre by perpendiculars from (xy) on them.

Hence the equation required must be

$$(x+g)(x_1+g)+(y+f)(y_1+f)=g^2+f^2-c$$

or $$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

(ii) Independent method.

Suppose (x_1y_1), (x_2y_2) to be two points on the circle. The equation

$$x^2+y^2+2gx+2fy+c=(x-x_1)(x-x_2)+(y-y_1)(y-y_2)$$

represents a straight line because the quadratic terms are the same on both sides.

It is true at the points (x_1y_1), (x_2y_2); for the left-hand side vanishes since these points are on the circle, and the right-hand side vanishes identically.

It must therefore represent the chord through these two points. Now put $x_2=x_1$, $y_2=y_1$, the chord becomes the tangent at (x_1y_1), and the equation becomes

$$x^2+y^2+2gx+2fy+c=(x-x_1)^2+(y-y_1)^2.$$

Bring all the terms to the left-hand side, we obtain

$$2xx_1+2yy_1+2gx+2fy-x_1^2-y_1^2+c=0.$$

But $$x_1^2+y_1^2+2gx_1+2fy_1+c=0,$$

therefore, substituting for $x_1^2+y_1^2$ and dividing by 2, the equation becomes

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

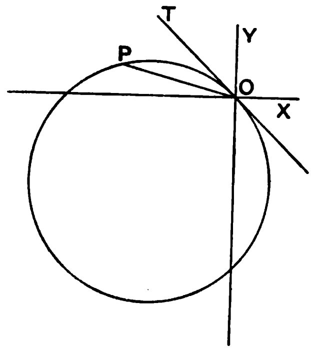
This equation is easily remembered: it is derived from the original equation by changing x^2 into xx_1, y^2 into yy_1, $2x$ into $x+x_1$ and $2y$ into $y+y_1$.

57. Suppose the constant term c to be zero, so that the circle passes through the origin.

The equation to the tangent at (0, 0) becomes

$$gx+fy=0.$$

For this is the equation to the straight line through the origin perpendicular to $\frac{x}{g} = \frac{y}{f}$ which is the equation to the radius.



We might have obtained this equation and that of the preceding article from the consideration that the tangent at any point meets the circle in a point indefinitely near the first.

Suppose P to be the point (xy), and $POX = \theta$, then the equation to OP is obviously $y = x \tan \theta$.

Let $OP = r$, then $x_1 = r \cos \theta$, $y_1 = r \sin \theta$.

Substitute these values in the equation to the circle; it becomes

$$r^2 + 2r(g \cos \theta + f \sin \theta) = 0,$$

a quadratic in r, such that *one* of the roots vanishes. This shews only that the line passes through the origin.

The other root is real unless $g \cos \theta + f \sin \theta = 0$, in which case both roots vanish, P coincides with O, and the chord becomes the tangent, whose equation is therefore

$$gx + fy = 0, \text{ as before.}$$

We can apply the same method to obtain the equation to the tangent at x_1y_1 by writing

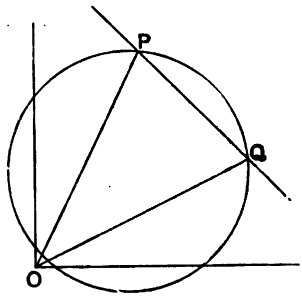
$$x_1 + r \cos \theta \text{ for } x,\ y_1 + r \sin \theta \text{ for } y$$

and finding the value of θ for which both values of r vanish.

The student should work out this on going through the subject a second time.

58. To find the condition that a straight line shall touch the circle. We might adopt either of the methods in Articles 52, 53, that is we might write down the condi-

tion that the perpendicular from the centre is equal to the radius, or that the quadratic obtained by eliminating y between the two equations shall have equal roots; we will however use another method which is not difficult, and has many advantages.



Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad (1)$$

and let the equation to the straight line be reduced to the form

$$lx + my = 1 \ldots\ldots (2).$$

Suppose the straight line to cut the circle in the points P and Q.

Then at P and Q both equations are true, and therefore

$$x^2 + y^2 + 2(gx + fy)(lx + my) + c(lx + my)^2 = 0 \ldots (3)$$

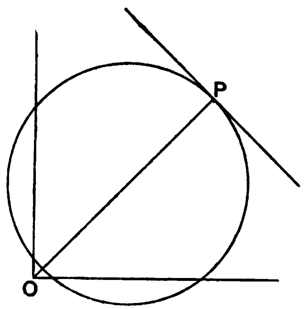
is true also.

But this equation is a *homogeneous quadratic*, and therefore represents two straight lines through the origin, which must be the lines OP, OQ. Rearrange this equation, it becomes

$$x^2(1 + 2gl + cl^2) + 2xy(gm + fl + clm) + y^2(1 + 2fm + cm^2) = 0.$$

If the straight line touches the circle, P and Q must coincide, and therefore OP, OQ also. The equation must be a perfect square, the condition for which is

$$(gm + fl + clm)^2$$
$$= (1 + 2gl + cl^2)(1 + 2fm + cm^2).$$



59. *To find the condition that two circles shall cut orthogonally.*

Circles are said to cut orthogonally when the angle between the tangents at the points of intersection is right.

If this is the case the radius of each circle at a point of intersection coincides with the tangent to the other circle.

Hence the radii are at right angles, and therefore the square on the distance between the centres is equal to the sum of the squares of the radii.

Now let the equations to the circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

The coordinates of the centres being

$$-g,\ -f;\ -g',\ -f',$$

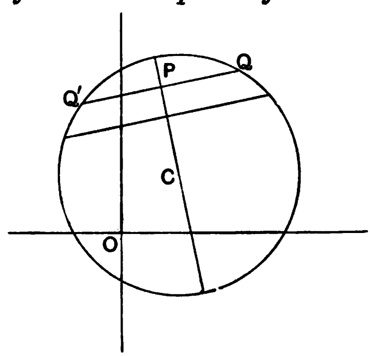
and the squares of the radii $g^2 + f^2 - c$, $g'^2 + f'^2 - c'$, respectively, the condition is

$$(g - g')^2 + (f - f')^2 = g^2 + f^2 - c + g'^2 + f'^2 - c',$$

or

$$2gg' + 2ff' = c + c'.$$

60. *To find the locus of the middle points of a series of parallel chords of a circle.*



Let $x'y'$ be the coordinates of *any* point P, xy of any point Q on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Suppose PQ to make an angle θ with the axis of x, and let $PQ = r$.

Then we know that

$$x = x' + r\cos\theta,\quad y = y' + r\sin\theta,\quad \text{(Art. 24)}$$

$$\therefore\ (x' + r\cos\theta)^2 + (y' + r\sin\theta)^2 + 2g(x' + r\cos\theta)$$
$$+ 2f(y' + r\sin\theta) + c = 0,$$

or $r^2 + 2r\{(x'+g)\cos\theta + (y'+f)\sin\theta\}$

$$+ x'^2 + y'^2 + 2gx' + 2fy' + c = 0.$$

Supposing we know x', y' and θ, this is a quadratic in r, the roots of which are the lengths of the portions of the chord which is drawn through P to cut the circle.

Now if P is the middle point of the chord $PQ = PQ'$, and these lines are drawn in opposite directions and therefore have opposite signs.

The values of r being equal and opposite, the coefficient of r in the quadratic must vanish.

Hence if $(x'y')$ is any point on the line

$$(x+g)\cos\theta + (y+f)\sin\theta = 0,$$

the chord through $x'y'$ which makes an angle θ with the axis of x is bisected at $x'y'$.

The equation represents a straight line through the centre $(-g-f)$, which is perpendicular to the bisected chords, as is evident from geometry.

61. To find an expression for the rectangle under the segments of a chord through a fixed point.

Take the equation of the preceding article, and let r_1, r_2 be the roots of the quadratic.

Then $\qquad r_1 r_2 = x'^2 + y'^2 + 2gx' + 2fy' + c.$

This expression is independent of θ, which shews that the rectangle under the segments of any chord through a fixed point is constant.

If the fixed point be without the circle, and $r_1 = r_2$, the expression given becomes the square of the tangent from the point to the circle.

Notice that the power of a point with respect to a given circle is the square of the tangent, real or imaginary, which can be drawn from that point to the circle.

EXAMPLES VI, *b*.

1. Write down the equations to the tangents at the origin, and at the extremities of the diameters through the origin, to the following circles:

$$x^2+y^2-3x+4y=0;$$
$$x^2+y^2+4x-2y=0;$$
$$x^2+y^2=ax;$$
$$x^2+y^2=by;$$
$$x^2+y^2=ax+by.$$

2. Prove that the circles and lines whose equations are here given touch each other, and find the points of contact in each case:

$$x^2+y^2+ax+by=0, \text{ and } ax+by+a^2+b^2=0;$$
$$x^2+y^2-2ax-2by+b^2=0, \text{ and } x=2a;$$
$$x^2+y^2-2c(x+y)+c^2=0, \text{ the axes and}$$
$$(x-c)\cos\theta+(y-c)\sin\theta=c;$$
$$x^2+y^2=ax+by, \text{ and } ax-by+b^2=0.$$

3. Find the condition that the straight line

$$x\cos\theta+y\sin\theta=p$$

should touch the circle $x^2+y^2=2(ax+by)$.

4. Find the equations to the tangents which are parallel to the axes in the following circles:

$$x^2+y^2=ax;$$
$$x^2+y^2+2by=0;$$
$$x^2+y^2+a^2=2a(x+y);$$
$$x^2+y^2-2ax-2by+a^2=0;$$
$$(x-h)^2+(y-k)^2=a^2;$$
$$x^2+y^2+4=4x-6y;$$
$$x^2+y^2+2y=2x+7.$$

5. In the circle $x^2+y^2=6x+8y$, give x integral values of the form $2m$, and write down the tangents at the points so found.

6. Find the values of a when $x+y=a$ touches the circle

$$x^2+y^2+2x-4y=3.$$

7. Find the values of a, b, and c when the lines

$$ax+3y=35,\ 3x+by=36,\ 4x-3y+c=0$$

touch the circle $\quad x^2+y^2=2x+4y+20.$

8. Find the equations to the circles of radius a, which touch the lines $x=y$, $x+y=0$.

9. The straight line $x+y=c$ cuts the circle

$$x^2+y^2=2(x+y)$$

in P and Q, find the value of c (i) when OPQ is an equilateral triangle; (ii) when P and Q coincide.

10. Find the equation to the lines joining the origin to the points of intersection of $x^2+y^2=8x+6y$ and $\frac{x}{4}+\frac{y}{3}=c$ and hence the value of c when the straight line touches the circle.

11. Find the equation to the lines joining the origin to the intersection of $x^2+y^2-2m(ax+by)+c=0$ and $\frac{x}{b}+\frac{y}{a}=1$, and the value of m when the line touches the circle.

12. Find the value of c when the straight line whose equation is $5x+12y=150$ touches the circle

$$x^2+y^2-2x+4y+c=0.$$

13. Find the equation to the circle which is inscribed in the triangle whose sides have for their equations

$$x=0,\ y=0,\ y=(x+a)\tan 2a.$$

14. Find the relation between g and f, when the straight line $ax+by=0$ touches the circle $x^2+y^2+2gx+2fy=0$.

15. What is the value of g when the straight line

$$3x-3y+2=0$$

touches the circle $\quad x^2+y^2+2gx-4y+6=0$?

16. Find the condition that $x\cos\alpha + y\sin\alpha = p$ should touch the circle $x^2 + y^2 + 2gx + 2fy = 0$.

17. The circles
$$x^2 + y^2 + 2ax + 2by = 0,$$
$$x^2 + y^2 + 2bx + 2ay = 0,$$
cut orthogonally.

18. If $\frac{x}{a} + \frac{y}{b} = 1$ touch $x^2 + y^2 + Ax + By + C = 0$,

then $$4\left\{\frac{A}{a} + \frac{B}{b} + 1 + C\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\} = \left(\frac{A}{b} - \frac{B}{a}\right)^2.$$

19. Find the equation to the circle which touches
$$Ax + By + C = 0$$
and has its centre at (h, k).

20. Find the equation to the circle which has its centre on the axis of x, and cuts $x^2 + y^2 = 9$, $5(x^2 + y^2) = 9x$, orthogonally.

21. Find the equation to the circle which passes through the points (a, b) $\left(-a, \frac{a^2}{b}\right)$ and touches the axis of x.

22. Prove that the straight line $r\cos(\theta - \alpha) = a$, touches the circle $r = a$ at the point a, α.

23. Prove that the equation to the tangent to
$$r = l\cos(\theta - \alpha),$$
at the point for which $\theta = \beta$, is
$$r\cos(\theta + \alpha - 2\beta) = l\cos^2(\beta - \alpha).$$

24. Find the equation to the tangents to the circle
$$r = l\cos(\theta - \alpha)$$
at the pole, and at the other extremity of the diameter through the pole.

25. Determine the points of contact of those tangents to the circle $r = l\cos(\theta - \alpha)$ which make an angle β with the initial line.

CHAPTER VII.

POLES AND POLARS. LOCI.

62. *From a given point tangents are drawn to a circle, to find the equation to the straight line passing through the points of contact.*

Let h, k be the coordinates of the given point, let x_1, y_1 be the coordinates of one point of contact, then the equation to the tangent at $(x_1 y_1)$ is

$$xx_1 + yy_1 = a^2;$$

but, since this passes through (h, k),

$$hx_1 + ky_1 = a^2.$$

Similarly, if x_2, y_2 be the coordinates of the other point of contact,

$$hx_2 + ky_2 = a^2,$$

since (h, k) lies on the tangent through $(x_2 y_2)$.

Hence, since
$$hx_1 + ky_1 = a^2,$$
$$hx_2 + ky_2 = a^2,$$

(x_1, y_1), (x_2, y_2) satisfy the equation

$$hx + ky = a^2;$$

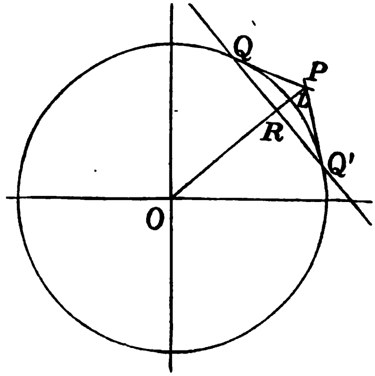
this, however, is the equation to a straight line, and since it is satisfied by the coordinates of two points, $(x_1 y_1)$, $(x_2 y_2)$, it is the equation of the straight line which passes

through those points: it is therefore the equation to the straight line required.

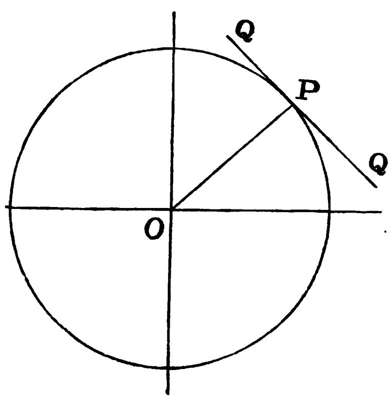
63. We have seen that when the point (h, k) is without the circle, the equation

$$hx + ky = d^2$$

represents the chord of contact of tangents through (h, k); when (h, k) is on the circle, the same equation represents the tangent at (h, k); what will then this equation represent when the point (h, k) is within the circle, so that no real tangents can be drawn through it to the circle?



It is still the equation to a straight line, and since its form is unchanged whatever be the *position* of the point (h, k) the equation must represent some geometrical facts which are independent of that *position*.



Let P be the point (h, k) either without, on, or within the circle, then the equation to OP is

$$\frac{x}{h} = \frac{y}{k} \quad \text{......(1)};$$

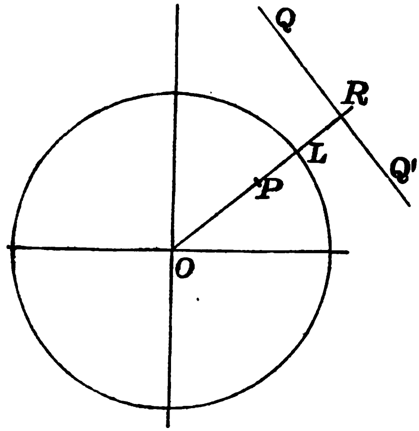
∴ (Art. 28), the line

$$hx + ky = d^2 \quad \text{....................} \quad (2)$$

is perpendicular to OP.

Again, the distance of the origin from the line (2) is

$\dfrac{d^2}{(h^2+k^2)^{\frac{1}{2}}}$, (Art. 20).



Now, $OP=(h^2+k^2)^{\frac{1}{2}}, OL=d$; therefore if in OP or OP produced we take a point R such that

$$OR : OL :: OL : OP,$$

and through R draw QRQ' perpendicular to OP, the equation to QRQ' will be

$$hx + ky = d^2.$$

QRQ' is called the *polar* of P: conversely, P is called the *pole* of QRQ'.

The polar of a point may be defined either geometrically or algebraically.

Geometrically, thus: let O be the centre of a circle, P any point, join OP and divide it, produced if necessary, in R, so that OR is a third proportional to OP and the radius: through R draw a straight line QRQ' at right angles to OP: this straight line is called the polar of P.

Conversely, let QQ' be any straight line, draw OR perpendicular to QQ', and in OR, produced if necessary, take a point P such that OP is a third proportional to OR and the radius, then P is called the pole of QQ'.

Algebraically: let the coordinates of any point be h, k; then the straight line represented by the equation

$$hx + ky = d^2$$

is called the polar of (h, k), with respect to the circle

$$x^2 + y^2 = d^2.$$

Conversely, let the equation to any straight line be thrown into the form

$$hx + ky = d^2,$$

then (h, k) is the pole of the line.

Thus; required the pole of

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Multiply by d^2,

$$\therefore \frac{d^2}{a}x + \frac{d^2}{b}y = d^2,$$

$$\therefore \frac{d^2}{a}, \frac{d^2}{b}, \text{ are the coordinates of the pole.}$$

The student is recommended to pay particular attention to the preceding articles; there is no part of the subject which it is more necessary to understand thoroughly.

64. *If Q lies on the polar of P, then the polar of Q passes through P.*

Let the coordinates of P be h, k, of Q, x', y', then since Q is on the polar of P, Q lies on the line

$$hx + ky = d^2 \text{.....................(1)},$$

$$\therefore hx' + ky' = d^2 \text{.....................(2)}.$$

But the polar of (x', y') is

$$xx' + yy' = d^2 \text{.....................(3)}.$$

In this equation if we put $x = h$, we get from (2), $y = k$.

Therefore the point (h, k) lies on the line (3), and conversely the line (3) passes through the point (h, k).

65. Similarly, if the polar of Q passes through P, then Q lies on the polar of P.

Let the coordinates of P be h, k, of Q, x', y', then the polar of Q is

$$xx' + yy' = d^2;$$

but, since this equation represents a straight line which passes through (h, k),

$$\therefore\ hx' + ky' = d^2;$$

$\therefore\ x', y'$, are the coordinates of a point which satisfies the condition

$$hx + ky = d^2;$$

that is, the point Q lies on the polar of P.

66. The intersection of two straight lines is the pole of the line joining their poles.

For suppose PQ, PR to be two straight lines, A and B their poles.

The polar of every point on PQ passes through A, and therefore the polar of P passes through A.

Similarly, the polar of P passes through B, that is, AB is the polar of P.

67. To find the polar of any point with respect to the circle whose equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Let $(x'y')$ be the point.

Then, just as before, the same equation will represent the polar whether $x'y'$ is without, on, or within the circle, that is whether the points of contact of tangents through $(x'y')$ are *real*, *coincident* or *imaginary*.

Suppose $(x_1y_1)\,(x_2y_2)$ to be these points, then (Art. 56)

$$x'x_1 + y'y_1 + g(x' + x_1) + f(y' + y_1) + c = 0,$$
$$x'x_2 + y'y_2 + g(x' + x_2) + f(y' + y_2) + c = 0.$$

Both (x_1y_1), (x_2y_2) are therefore on the line

$$xx' + yy' + g(x + x') + f(y + y') + c = 0,$$

which is therefore the equation required.

Cor. The polar of the origin is

$$gx + fy + c = 0.$$

EXAMPLES VII, *a*.

The first ten questions refer to the circle whose equation is

$$x^2 + y^2 = d^2.$$

1. Write down the polars of the following points:

$$(d, d),\ (3d, -2d),\ (a, 0),\ (0, -b).$$

2. Find the poles of the lines

$$x = a,\ y = b,\ Ax + By + C = 0,\ \frac{x}{a} + \frac{y}{b} = 1.$$

3. If the pole lie on the circle $x^2 + y^2 = 4d^2$, the polar touches the circle

$$x^2 + y^2 = \frac{d^2}{4}.$$

4. If the polar touch the circle $x^2 + y^2 = m^2d^2$, the pole is on the circle

$$x^2 + y^2 = \frac{d^2}{m^2}.$$

5. The pole of $y = mx + c$, where m is variable, lies on the line $cy = d^2$.

6. The pole of $y = mx + c$, where c is variable, lies on the line $my + x = 0$.

7. The pole of $my = x - a$, where m is variable, lies on the line $ax = d^2$.

8. The pole of $3x - 4y + d = k(5x - 8y + 3d)$, where k is variable, lies on the line $x + y = d$.

9. If the pole is on the line $\frac{x}{a} + \frac{y}{b} = 1$, the equation to the polar may be written in the form

$$ax - d^2 = m(by - d^2).$$

10. The straight lines $x = h$, $x = k$, cut the circle in A, B, C, D respectively: AC, BD intersect in E, AD, BC in F; shew that E, F are conjugate points, (*i.e.*) that the polar of each passes through the other.

11. Write down the polars of (2, 8), (8, 2) with respect to the circle $x^2 + y^2 = 25$, and verify the theorem that the line joining the 2 points is the polar of their intersection of their polars.

12. A, B, C, D are the points $(-5, 0)$; $(0, 5)$; $(4, 3)$, $(5, 0)$ on the circle $x^2 + y^2 = 25$; AC, BD intersect in E, AD, BC in F, AB, CD in G, shew that the triangle EFG is such that each side is the polar of the intersection of the others.

N.B. Such triangles are called self-conjugate: this example is a special case of a general theorem, which is true if A, B, C, D are any points on a circle.

13. Verify the above theorem, when the circle is the same, and the coordinates of the 4 points are $(-3, -4)$, $(0, 5)$, $(4, 3)$, $(4, -3)$ respectively.

14. The circle whose equation is

$$x^2 + y^2 - 10x - 11y + 24 = 0$$

cuts the axis of x in A, B, and that of y in C, D respectively; AD, BC meet in E; AC, BD in F; shew that EF is the polar of the origin.

15. Find the polar of (g, f) with respect to each of the circles

$$x^2 + y^2 + 2gx + 2fy = 0, \text{ and } x^2 + y^2 + 2gx + 2fy + c = 0.$$

16. Find the pole of $\frac{x}{a} + \frac{y}{b} = 1$ with respect to the circle

$$x^2 + y^2 + 2gx + 2fy = 0.$$

17. If (a, b) be the pole of $lx + my = \delta$ with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

then
$$\frac{a+g}{l} = \frac{b+f}{m} = \frac{ag+bf+c}{\delta}.$$

18. In the circles

$$x^2 + y^2 = 16, \ x^2 + y^2 - 10x + 16 = 0,$$

the centre of each is the polar of the common chord with respect to the other.

19. In the circles

$$x^2 + y^2 = 9, \quad x^2 + y^2 - 10x + 9 = 0,$$

if CD be a diameter of either, C is on the polar of D, and D on that of C with respect to the other.

20. Two circles are such that the polar of the centre of each with respect to the other is the same: shew that this polar is the common chord of the two, and that they cut orthogonally.

Take the circles as

$$x^2 + y^2 = d^2,\ (x-a)^2 + y^2 = r^2.$$

21. If the circles $x^2 + y^2 = d^2$, $(x-a)^2 + y^2 = r^2$, cut orthogonally, then if CD be a diameter of either, C is on the polar of D, and D on that of C with respect to the other.

LOCI.

68. We are often required to find the locus of a point which moves subject to some given law: no general rule can be given for finding the equation to such a locus; it generally however results from elimination between two or more equations.

Sometimes the mere algebraic statement of the question as in (i) leads to the equation, more often as in (ii) the point is the intersection of two straight lines which involve a quantity to be eliminated.

We will give a few examples.

The figures are simple, and it will be useful to the student to draw them for himself.

(i) To find the locus of a point the distances of which from two given points are in a constant ratio.

Let O, A be the two given points, P a point on the locus.

Take O as origin, OA as axis of x; let $OA = a$, and let the coordinates of P be (x, y): let $OP = mAP$.

Now $\quad OP = (x^2 + y^2)^{\frac{1}{2}},\ AP = \{(x-a)^2 + y^2\}^{\frac{1}{2}};$

$$\therefore\ x^2 + y^2 = m^2\{(x-a)^2 + y^2\},$$

$$(1 - m^2)(x^2 + y^2) + 2am^2x - m^2a^2 = 0.$$

The locus is therefore (Art. 45) a circle, of which the centre is on the axis of x.

(ii) To find the locus of the intersection of two straight lines, which pass each through a given point and contain a given angle. (Of course we know from geometry that this

locus is a circle, we will however obtain this result analytically.)

Let A, B be the given points, and let $AB = 2a$.

Let P be a point on the locus, and $APB = \alpha$.

Take the middle point of AB as origin, AB as axis of x.

Let AP make an angle θ with the axis of x.

Then its equation is (Art. 23)

$$y = (x - a) \tan \theta \text{ ....(1).}$$

Let BP make an angle ϕ with the axis, then its equation is

$$y = (x + a) \tan \phi \text{ ....(2).}$$

But $\theta - \phi = \alpha$;

$$\therefore \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \tan \alpha.$$

Now, from (1) and (2),

$$\tan \theta = \frac{y}{x - a}, \quad \tan \phi = \frac{y}{x + a};$$

$$\therefore \frac{y(x + a) - y(x - a)}{(x^2 - a^2) + y^2} = \tan \alpha;$$

$$\therefore x^2 + y^2 - 2ay \cot \alpha = a^2,$$

the equation to the circle whose centre is on the axis of y at a distance $a \cot \alpha$ from the origin, and radius $a \operatorname{cosec} \alpha$.

(iii) ABC is a triangle, P a point such that the sum of its distances from the sides is constant; find the locus of P.

Let the sum of the distances be c, and the equations to the sides

$$x \cos \alpha + y \sin \alpha = p,$$
$$x \cos \beta + y \sin \beta = q,$$
$$x \cos \gamma + y \sin \gamma = r.$$

Then if (x, y) be the coordinates of P, the distances of P from the sides are

$x \cos \alpha + y \sin \alpha - p$, $x \cos \beta + y \sin \beta - q$, $x \cos \gamma + y \sin \gamma - r$,
respectively.

$\therefore$ $x(\cos \alpha + \cos \beta + \cos \gamma) + y(\sin \alpha + \sin \beta + \sin \gamma) = p + q + r + c$,
the equation to a straight line.

(iv) C is a fixed point, and through C a straight line is drawn to cut a fixed circle in P and Q; find the locus of a

point R on this line such that CR is (α) an arithmetic, (β) a geometric, (γ) a harmonic mean between CP and CQ.

Take the centre of the given circle as the origin and let its equation be

$$x^2 + y^2 = d^2 \text{ ....(1).}$$

Let CPQ make an angle θ with the axis of x, and let the coordinates of C be h, k: then the equation to CQ may be written

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r, \text{ (Art. 24)..............(2),}$$

where r is the distance between the points (xy), (hk).

Substituting the values of x and y obtained from equations (2) in (1), we obtain a quadratic in r, which gives the lengths of CP and CQ.

Rearranging this quadratic we get

$$r^2 + 2r(h\cos\theta + k\sin\theta) + h^2 + k^2 - d^2 = 0 \text{ ...... (3).}$$

If r_1, r_2 be the roots of this quadratic, and (x, y) be now the coordinates of R the point in question, and if $CR = \rho$; then, since R is on (2),

$$x = h + \rho\cos\theta,\ y = k + \rho\sin\theta.$$

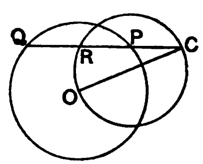
Now, from (3),

$$r_1 + r_2 = -2(h\cos\theta + k\sin\theta),\ r_1r_2 = h^2 + k^2 - d^2.$$

Then, (α) if ρ be an arithmetic mean between r_1 and r_2,

$$\rho = \frac{r_1 + r_2}{2} = -(h\cos\theta + k\sin\theta);$$

$$\therefore\ \rho^2 + h\rho\cos\theta + k\rho\sin\theta = 0,$$



or

$$(x-h)^2 + (y-k)^2 + h(x-h) + k(y-k) = 0,$$

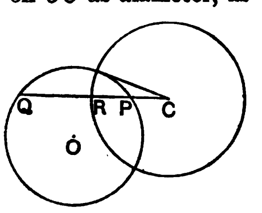
or

$$x^2 + y^2 = hx + ky.$$

This equation represents the circle on OC as diameter, as is evident from geometry.

(β) If ρ be a geometric mean between r_1 and r_2,

$$\rho^2 = r_1r_2 \text{ or } (x-h)^2 + (y-k)^2 = h^2 + k^2 - d^2.$$



Now $h^2 + k^2 - d^2$ is the square on the tangent from C to the circle, hence the equation represents the

circle whose centre is C, and radius the tangent from C to the circle.

(γ) If ρ be a harmonic mean between r_1 and r_2,

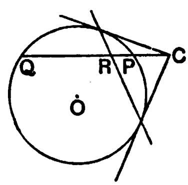
$$\frac{2}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{r_1+r_2}{r_1 r_2};$$

$$\therefore\ 2r_1r_2=(r_1+r_2)\rho;$$

$$\therefore\ h^2+k^2-d^2+(h\cos\theta+k\sin\theta)\rho=0,$$

$$h^2+k^2-d^2+h(x-h)+k(y-k)=0,$$

$$hx+ky=d^2,$$



the polar of C.

Hence we get this important theorem: if a straight line through a point C cut a circle in P, Q and CP, CR, CQ are in harmonic progression, the locus of R is a straight line.

This is sometimes taken as the definition of a polar and the theorems of the preceding part of the chapter deduced from it.

All these loci must pass through the points where tangents from C meet the circle, since at these points the values of r become equal, and therefore their means coincide with those equal values.

Portions of these loci are, however, without the circle, can we interpret the equations in these cases?

If θ in (2) have such a value that the line does not meet the circle, the values of r_1, r_2 which are the roots of (3) are imaginary.

Their sum and product are, however, real, and so therefore their means, and so we arrive at the anomaly of impossible points lying on a real line, and being such that the point of bisection of the line joining them is real.

This anomaly arises from the fact that in symbolical algebra we can attach a meaning to the impossible roots of equations while we are unable to interpret these expressions geometrically.

Our algebra is therefore more general than our geometry.

69. In treating problems on loci, the choice of axes is very important, as the equations which result from the statement of the questions may be simple or the reverse as appropriate axes are chosen.

If a fixed point is given, it should generally be taken as the origin.

If a straight line of finite length is given, it is best to denote it by $2a$, and to consider it as the axis of x, the origin being the middle point.

If the origin is otherwise chosen, take the axis of x perpendicular to the given line, the equation to which may be written $x = a$.

If two straight lines inclined at a given angle are given, take their point of intersection as the origin, and the axes bisecting the supplementary angles between them, the lines will then be written in the forms

$$y = x \tan \alpha, \quad y = -x \tan \alpha.$$

If a circle is involved and it is inconvenient to take the centre as origin, it is generally best to take one of the axes as passing through the centre, and to use the equation

$$(x - a)^2 + y^2 = d^2.$$

Polar coordinates are very useful when we have to discuss lines drawn from a fixed point: at any point in the proof we may substitute $r \cos \theta$, $r \sin \theta$ for x and y; but if we have to substitute for r, we must be careful to square in case the single power r occurs, without having $\sin \theta$ or $\cos \theta$ as a factor.

Examples VII, *b*.

1. The locus of a point, the algebraic sum of whose distances from the sides of a given polygon is constant, is a straight line.

2. The locus of a point, the distances of which from two fixed straight lines are in a given ratio, is a straight line.

3. O is a fixed point, OPQ a straight line, find the locus of Q, when

(i) $OQ = mOP$, and P moves on a fixed straight line.

(ii) $OQ = mOP$, and P moves on a fixed circle.

(iii) $OP \,.\, OQ$ is constant, and P moves on a fixed straight line.

(iv) $OP \,.\, OQ$ is constant and P moves on a fixed circle through O.

(v) $OP \,.\, OQ$ is constant and P moves on a fixed circle which does not pass through O.

N.B. If $OP \,.\, OQ$ is constant, Q is said to be the inverse of P.

4. ABC is a triangle, the base BC is fixed, and of length $2a$; find the locus of A, when

(i) $AB^2 - AC^2 = a^2$,

(ii) $AB = 2AC$,

(iii) $AB^2 + AC^2 = c^2$,

(iv) $\tan ABC = m \tan ACB$.

5. ABC is a fixed triangle, DE a straight line parallel to BC, cutting the sides at D and E respectively; if ADP, AEP are right angles find the locus of P.

6. In the preceding question, if DPF, EPG are drawn at right angles to the sides AC, AB respectively, find the locus of P.

7. A, B are two fixed points on the axis of x, C, D on that of y, find the locus of P, when

(i) $\Delta APB + \Delta CPD = a^2$,

(ii) $\Delta APB - \Delta CPD = a^2$,

(iii) $m\Delta APB + n\Delta CPD = a^2$.

8. ABC is a triangle whose angles are given: if A be fixed and B move along a straight line, C moves along another straight line.

9. A, B, C, D are fixed points on a straight line: if P be a point such that $\angle APB = \angle CPD$, find the locus of P.

10. AB is a given diameter of a circle of which C is the centre, DE a chord such that $DCE = 2a$: AD, BE produced intersect in F, shew that the locus of F is a circle. If AE, BD intersect in G, the locus of G is also a circle.

11. Find the locus of a point P, such that its distance from $x \cos \alpha + y \sin \alpha = p$ is a third proportional to a given line b, and its distance from (h, k).

12. If B and C move on two parallel lines, and AB, AC be inclined to these lines at angles α, β respectively, find the locus of A, if $AB = mAC$.

13. ACB, DCE are two straight lines of fixed length which intersect in the middle point of AB: $ACD = \alpha$, and C moves along DE; if AD meet BE in F, find the locus of F.

14. ABC is a triangle, DE a straight line parallel to BC cutting the sides AB, AC in D, E respectively; if BE, CD intersect in F, find the locus of F.

15. OAB, OCD are two fixed straight lines; A, B, C, D being fixed points: PAB, PCD are two triangles: shew that if $mPAB + nPCD = a^2$, the locus of P is a straight line.

CHAPTER VIII.

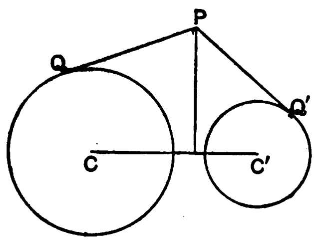
RADICAL AXES. CO-AXAL CIRCLES.

70. *The locus of a point from which equal tangents can be drawn to two circles is a straight line.*

Let P be a point (xy), and

$$x^2+y^2+2gx+2fy+c=0$$

the equation to a circle.



Then (Art. 61)

$$x^2+y^2+2gx+2fy+c$$

is the square on the tangent from P to the circle.

So if

$$x^2+y^2+2g'x+2f'y+c'=0$$

be the equation to a second circle,

$$x^2+y^2+2g'x+2f'y+c'$$

is the square on the tangent from P to it.

If these tangents are equal we must have

$$x^2+y^2+2gx+2fy+c=x^2+y^2+2g'x+2f'y+c'$$

or $$2(g-g')x+2(f-f')y+c-c'=0.$$

But this is the equation to a straight line, which proves the proposition.

This straight line is called the *radical axis* of the two circles.

Let us denote the expressions

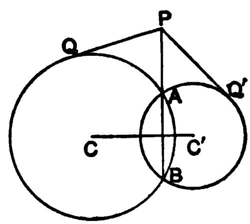
$$x^2+y^2+2gx+2fy+c,\ x^2+y^2+2g'x+2f'y+c'$$

by S, S' respectively, then the equation to the radical axis is

$$S-S'=0.$$

If the circles cut one another, S and S' are each zero at the common points, and therefore the radical axis passes through those points.

In this case, part of the radical axis is within the circles as in the figure, and so no tangents can be drawn to either circle.



S and S' are however equal to the product of the segments of any straight line through a point in the chord cutting the circles, and so we may extend our definition thus.

If P be a point and two straight lines through P cut the circles in Q, R; Q', R' respectively then if

$$PQ \,.\, PR = PQ' \,.\, PR',$$

the locus of P is a straight line called the radical axis.

If the two circles touch, the radical axis is the common tangent at the common point.

This gives us the easiest test whether two circles touch one another.

Find the equation to the radical axis, and write down the length of the perpendicular from the centre of either circle on it.

If this perpendicular is equal to the radius of the corresponding circle, the two circles touch one another.

71. *The radical axis is perpendicular to the line joining the centres of the circles.*

The centres C, C' are the points $(-g, -f)$ $(-g' -f')$ and therefore the line CC' makes an angle

$$\tan^{-1}\frac{f-f'}{g-g'},$$

with the axis of x.

Now the radical axis makes

$$\tan^{-1}\left(-\frac{g-g'}{f-f'}\right)$$

with the same axis.

Hence these straight lines are at right angles.

72. *Radical centre.*

Suppose we have a third circle

$$S'' \equiv x^2 + y^2 + 2g''x + 2f''x + c'' = 0.$$

Then the radical axis of $S = 0$ and $S' = 0$ is $S = S'$.

Similarly that of $S = 0$ and $S'' = 0$ is $S = S''$.

Therefore at their intersection $S' = S''$, which is the equation to the radical axis of S' and S''.

The radical axes of three circles taken in pairs therefore intersect in a point.

This point is called the *radical centre* of the circles.

73. *Co-axal circles.*

Let $S = 0$, $S' = 0$ be the equations to two circles where S, S' have the same meanings as before, then $S + kS' = 0$ represents a circle which passes through the points of intersection, *real* or *imaginary* of S and S'.

Circles are said to cut in imaginary points when they havè no real points of intersection, since in this case the algebraic solution of the two equations gives imaginary values of x and y.

These imaginary values satisfy the equations $S = 0$, $S' = 0$, and therefore $S = S'$ which is the equation to the radical axis.

Now $S+kS'=0$ represents a circle which passes through these points real or imaginary, and which therefore has the same radical axis as $S=0$, and $S'=0$.

All circles whose equation is $S+kS'=0$ have therefore the same radical axis and are called coaxal circles.

The student may have found the preceding reasoning a little difficult to follow, so we give an alternative proof.

The equation

$$x^2+y^2+2gx+2fy+c+k(x^2+y^2+2g'x+2f'y+c')=0$$

may be written

$$x^2+y^2+\frac{2(g+kg')}{1+k}x+\frac{2(f+kf')}{1+k}y+\frac{c+kc'}{1+c}=0$$

and therefore represents a circle.

Now the radical axis of this circle and $S=0$ is

$$2gx+2fy+c=\frac{2(g+kg')}{1+k}x+\frac{2(f+kf')}{1+k}y+\frac{c+kc'}{1+k}$$

or $$k(2gx+2fy+c)=k(2g'x+2f'y+c')$$

but this is the equation to the radical axis of S and S'.

Q. E. D.

74. Since the radical axis of any pair of coaxal circles is perpendicular to the line joining the centres, suppose A, B, C to be the centres of three circles which have the same radical axis.

Then since that axis is perpendicular to AB and AC, A, B, C are in the same straight line.

In order to study the properties of coaxal circles, it is most convenient to take the line of centres as the axis of x, and the radical axis as that of y.

The general equation reduces to

$$x^2+y^2+2gx+c=0.$$

There are two cases.

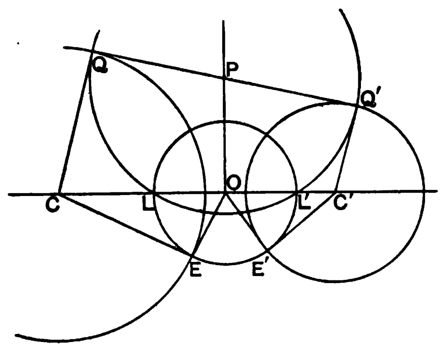
(i) Let c be positive, then it is the square of the tangent from the origin to one of the circles.

But since the origin is on the radical axis, the tangents from it to all the circles are equal.

Put $c = l^2$, then $x^2 + y^2 + 2gx + l^2 = 0$, where l is the same for all circles of the system, but g is variable, will represent any circle of a system. The radius of any circle is $\sqrt{g^2 - l^2}$, from which we see that $g > l$.

If $g^2 = l^2$, the radius of the circle vanishes, and the circle becomes one of two points on the axis of x, at distances l on each side of the origin.

The points so obtained are called the limiting points of the system, and may be regarded as infinitely small circles belonging to the system. We will discuss the system by means of a figure.



Let O be the origin, take OL, OL' each equal to l, on the axis: with O as centre, OL as radius describe a circle LEL'.

Take any point C on the axis of x on OL or OL' produced and let $OC = g$.

Draw OE to touch the circle LEL', then the circle with centre C and radius CE is one of the circles of the system.

For the coordinates of its centre are $-g$, 0, and the radius CE or $(g^2 - l^2)^{\frac{1}{2}}$, and its equation is

$$x^2 + y^2 + 2gx + l^2 = 0,$$

an equation which, as we have seen, represents a circle of the system.

We can verify the property of the radical axis very easily. Suppose we take any point P on the axis of y, and let $OP = b$. Let PQ be a tangent from O to this circle.

Then the tangent from $0, b$ to the circle

$$x^2 + y^2 + 2gx + l^2 = 0 \text{ is } \sqrt{b^2 + l^2}$$

which is independent of g, and therefore the same for all circles of the system.

The equation to the circle whose centre is P and radius PQ is

$$x^2 + (y - b)^2 = b^2 + l^2,$$

or
$$x^2 + y^2 - 2by - l^2 = 0.$$

If we put $y = 0$ in this equation we get $x = \pm l$.

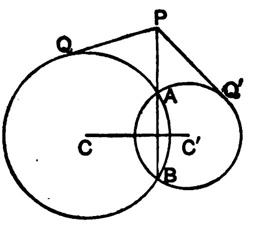
This circle passes through L and L', and belongs to another system of coaxal circles, which intersect at L, L' and have the axis of x for their radial axis.

It is easy to see (Art. 59) that every circle of one system cuts every circle of the other orthogonally.

This is obvious geometrically since PQ which is the tangent to one circle is the radius of the other.

(ii) Let the circles intersect; then c is negative.

Since the radical axis is the axis of y, the circles must all pass through the same points on that axis at equal distances from the origin.

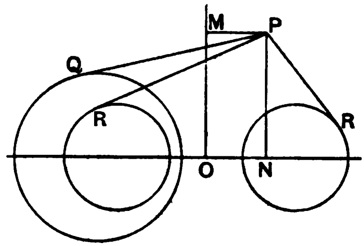


The equation to any of these circles will therefore be obtained by changing the sign of l^2, and will be

$$x^2 + y^2 + 2gx - l^2 = 0.$$

In this case there are no limiting points, as the smallest circle of the system is that on the common chord as diameter.

75. The difference between the squares on the tangents from a point to two circles is proportional to the distance of the point from the radical axis.



Let the equations to the circles be

$x^2 + y^2 + 2gx + l^2 = 0,$

$x^2 + y^2 + 2g'x + l^2 = 0,$

so that the radical axis is the axis of y.

Let P be the point (xy), PQ, PR the tangents from P.

Draw PN, PM perpendicular to the axes.

Then $$PQ^2 = x^2 + y^2 + 2gx + l^2$$

$$PR^2 = x^2 + y^2 + 2g'x + l^2$$

$$\therefore PQ^2 - PR^2 = 2(g - g')x.$$

But $$PM = ON = x$$

$\therefore PQ^2 - PR^2$ is proportional to PM.

If R and Q are on opposite sides of OM, the equation to the circle on which R lies will be

$$x^2 + y^2 - 2g'x + l^2 = 0,$$

and then we shall have

$$PQ^2 - PR^2 = 2(g + g')x.$$

In these equations l^2 may be negative, that is the circles may cut one another.

Examples VIII.

1. Write down the equations to the radical axes of the following pairs of circles:

(i) $x^2 + y^2 = 9$, and $x^2 + y^2 + 2x - 3y + 7 = 0$:

(ii) $x^2 + y^2 = 2ax$, and $x^2 + y^2 = 2by$:

(iii) $x^2 + y^2 - 6x + 10y + 30 = 0$, and $x^2 + y^2 - 6x + 8y = 5$.

2. Shew that the following pairs of circles touch one another, and find the point of contact for each pair:

(i) $x^2+y^2=25$, and $x^2+y^2=3x+4y$:

(ii) $x^2+y^2=a^2+b^2$, and $x^2+y^2+ax+by=0$:

(iii) $x^2+y^2=24x+10y$, and $x^2+y^2+1014=60x+25y$.

3. Shew that all circles which have the same constant term in their equations have the origin for radical centre.

4. Find the radical axes and radical centre of the circles

$$x^2+y^2-2x-4y-1=0,$$
$$x^2+y^2-4x-6y+5=0,$$
$$x^2+y^2-6x-2y=0.$$

5. The radical axis bisects the common tangents to two circles.

6. Find an equation which involves an indeterminate constant m, and which will represent all circles which have the same radical axis as

$$x^2+y^2=4,\ x^2+y^2+8=6\,(x+y).$$

Find the equation to the smallest of these circles.

7. The abscissæ of the centres of two circles are a_1, a_2 and the lengths of the tangents from the origin l_1, l_2 respectively: shew that the radical axis cuts the axis of x at the point for which

$$x=\frac{l_1^2-l_2^2}{2(a_1-a_2)}.$$

8. A fixed circle is cut by a series of circles, all of which pass through two fixed points, shew that the radical centre is a fixed point.

9. Three circles have fixed centres, and their radii are $r_1+\rho$, $r_2+\rho$, $r_3+\rho$, where ρ is variable: shew that their radical centre lies on a fixed straight line.

10. If a series of circles be such that the polar of a fixed point with reference to any one of them is a fixed straight line, they will have a common radical axis.

CHAPTER IX.

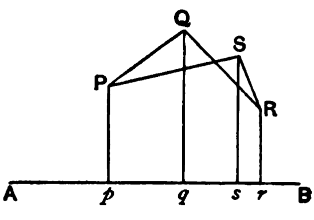
PROJECTIONS, OBLIQUE AXES, TRANSFORMATIONS.

76. WE propose in this chapter to give a short account of some methods, with which the student should be acquainted, though they are not of the same importance as the methods of rectangular and polar coordinates which we have hitherto used.

77. *Projection.*

Suppose we have any points P, Q, R, S in a plane, and a straight line AB.

From P, Q, R, S draw perpendiculars Pp, Qq, Rr, Ss on AB; then p, q, r, s are called the *projections* (or more strictly the *orthogonal projections*) of P, Q, R, S and pq, qr, rs the projections of the lines PQ, QR, RS on AB.



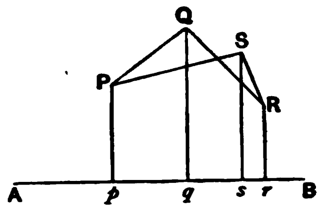
The convention with regard to sign which we have hitherto used, that if a line measured in one direction is positive, one measured in the opposite direction is negative, applies to projections: thus in the figure, rs is negative. If α be the angle between a straight line and its projection and l be the length of the line, it is clear that $l \cos \alpha$ is the length of the projection. In the figure the angle RS

makes with its projection is obtuse: the student must be careful to take the letters in the proper order: thus in the figure the projection of RS is rs and is negative, but that of SR is positive.

Suppose a point to move from P to S, first along the straight line PQ, then along QR, and finally along RS, the projections of its path will be $pq + qr + rs$

or $pq + qr - sr$,

which is that of the line PS.



The algebraic sum of the projections of the several parts of a broken line is therefore the projection of the line joining the extremities.

Hence it follows that the projection of a closed figure is always zero.

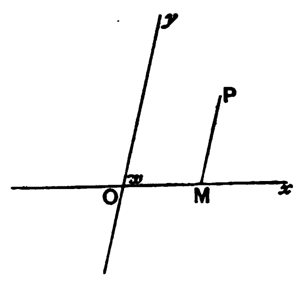
The method of projection is of much more utility in solid than in plane geometry.

If a figure bounded by straight lines or curves is drawn, and from every point in its perimeter perpendiculars are drawn on a plane thus forming another figure: the second is said to be the projection of the first.

78. *Oblique axes.*

Suppose the angle between the axes OX, OY not to be a right angle.

From P draw PM parallel to OY to cut OX in M, then OM, PM are called the oblique coordinates of P, belonging to the system OX, OY.



It is obvious that if OM, PM are known, P is known also.

The angle XOY is generally denoted by ω.

79. The methods of oblique coordinates generally lead to more complicated results than those we have arrived at by other methods, and should never be used where the magnitudes of angles are involved.

It is obvious that where an investigation does not depend on the value of the angle between two straight lines, the results will be the same whatever that angle may be, but that any results derived from Euclid I. 47 will have to be modified if the angle between the axes is no longer a right angle.

We will give a list of the articles which are not altered when the axes become oblique, and will discuss the most important modifications which have to be made in that case.

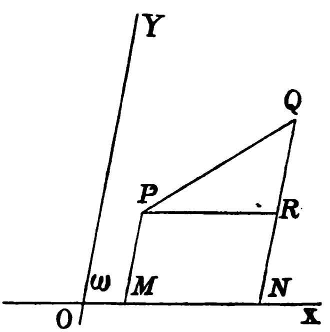
80. *In chapter I.*

Art. 8 is unaltered.

In Art. 9 the expression for the area of the triangle must be multiplied by $\sin\omega$.

The following must be substituted for Arts. 6, 7.

To find the distance between the points (x_1y_1) (x_2y_2) when the angle between the axes is ω.



Let OM, PM, ON, QN be the coordinates of P, Q: draw PR parallel to OX cutting QN at R: then by trigonometry

$$PQ^2 = PR^2 + QR^2 - 2PR \, . \, QR \cos PRQ,$$

but

$$PRQ = ONQ = \pi - \omega;$$

$$\therefore PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega.$$

Hence, or independently, the distance of (x_1y_1) from the origin is

$$\{x_1^2 + y_1^2 + 2x_1y_1\cos\omega\}^{\frac{1}{2}}.$$

81. *Analysis of chapter II.*

The discussion of points on a locus does not depend on the angle between the axes, except in the case of the circle on p. 16.

The proofs and results of Arts. 16, 17, 18, 19, 21, 22, are unaltered for oblique coordinates, the lines which are perpendicular to the axis of x being now parallel to that of y.

Thus the equation $lx + my = d$ always represents a straight line, whose intercepts on the axes are $\frac{d}{l}$, $\frac{d}{m}$ respectively.

In this equation if l and m are of the same sign, the line makes an angle $> \omega$ with the axis of x, and $< \omega$ if l and m are of opposite signs.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

still represents the straight line through (x_1y_1) (x_2y_2), but $\frac{y_2 - y_1}{x_2 - x_1}$ is no longer the tangent of the angle it makes with the axis of x.

Arts. 20, 23, 24, 25 are completely altered: we will discuss the alterations and those in chapters III. and IV. when we have finished the enumeration of those articles which are unaltered.

82. The proofs of *chapter III.* depend almost entirely on the axes being rectangular, the only exception being in the case of parallel straight lines.

83. *In chapter IV.*, the conditions that three points may be on the same line, or that three straight lines may pass through the same point, do not depend on the angle between the axes.

The equations

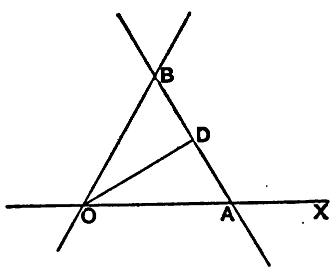
$$lx + my - d = k(l'x + m'y - d'), \ ax^2 + 2hxy + by^2 = 0$$

have the same meaning as before, and the condition that the general equation shall represent two straight lines is also unaltered, but the reasoning of the rest of the chapter is based on the perpendicularity of certain straight lines and is therefore inapplicable to oblique coordinates.

84. The discussion of the circle depends almost entirely on the square of the distance of (xy) from the origin being $x^2 + y^2$, which is not the case when the axes are oblique.

85. We will now discuss the most important theorems about the straight line and circle when the axes are inclined at an angle ω.

86. *To find the angle which the line $lx + my = d$ makes with the axis of x, and distance from the origin.*



With the construction of Art. 20, AOB being ω,

$$\frac{\sin OAB}{\sin OBA} = \frac{\sin(\pi - \theta)}{\sin(\theta - \omega)} = \frac{-\sin\theta}{\sin(\omega - \theta)}.$$

But

$$\frac{\sin OAB}{\sin OBA} = \frac{OB}{OA} = \frac{l}{m}:$$

$$\therefore\ m\sin\theta = -l\sin(\omega - \theta) = -l(\sin\omega\cos\theta - \cos\omega\sin\theta),$$

$$\therefore\ \tan\theta = \frac{l\sin\omega}{l\cos\omega - m}.$$

Again, $OD \,.\, AB = OA \,.\, OB\sin\omega$, each being double the area of the triangle AOB.

But

$$OA = \frac{l}{d},\ OB = \frac{d}{m},\ \text{and}\ AB^2 = OA^2 + OB^2 - 2OA\,.\,OB\cos\omega$$

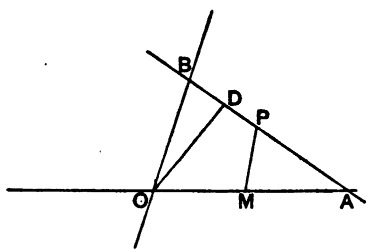
$$\therefore\ OD = \frac{d\sin\omega}{(l^2 + m^2 - 2lm\cos\omega)^{\frac{1}{2}}}.$$

Just as in article 30, the perpendicular distance of (x_1y_1) from this line is

$$\frac{(d - lx_1 - my_1)\sin\omega}{(l^2 + m^2 - 2lm\cos\omega)^{\frac{1}{2}}}.$$

In these expressions if we put $\omega = \frac{\pi}{2}$, we get the corresponding expressions for rectangular coordinates, as manifestly ought to be the case.

87. To find the equation to a straight line in terms of the perpendicular from the origin, and the angles it makes with the axis of x.



Let AB be the straight line, OD the perpendicular from O, let $OD = p$, $DOA = \alpha$, $DOB = \beta$, so that

$$\alpha + \beta = \omega.$$

Then OD is the sum of the projections of OM, MP on OD. (Art. 77.)

$\therefore$ $x\cos\alpha + y\cos\beta = p$ is the equation required.

By making $\omega = \frac{\pi}{2}$, we get the equation

$$x\cos\alpha + y\sin\alpha = p$$

in rectangular coordinates: this method is easier than that given in Art. 25.

88. It is easy to see from Art. 86, or from an independent investigation, that if the equation to the straight line be given in the form $y = mx + c$, $m = \frac{\sin\theta}{\sin(\omega - \theta)}$ where θ is the angle the line makes with the axis of x, and hence that the angle between

$y = mx + c$, $y = m'x + c'$ is

$$\tan^{-1}\frac{(m - m')\sin\omega}{1 + mm' + (m + m')\cos\omega}.$$

These straight lines are therefore parallel if $m = m'$ and perpendicular if $1 + mm' + (m + m') \cos \omega = 0$.

89. *To find the equation to the circle in oblique coordinates.*

Since the distance of (xy) from the origin and from the point (ab) are $(x^2 + y^2 + 2xy \cos \omega)^{\frac{1}{2}}$
and $\{(x - a)^2 + (y - b)^2 + 2(x - a)(y - b) \cos \omega\}^{\frac{1}{2}}$
respectively, it follows that the equations

$$x^2 + y^2 + 2xy \cos \omega = d^2,$$

$$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b) \cos \omega = d^2,$$

represent the circles whose radius is d, and centres the origin and the point (a, b) respectively.

The general equation becomes in this case

$$x^2 + 2xy \cos \omega + y^2 + 2gx + 2fy + c = 0.$$

The equations to the tangent at (x_1y_1) to these circles are very rarely used; the methods of Arts. 49, 56 may be employed.

The equations will be found to be

$$xx_1 + (xy_1 + x_1y) \cos \omega + yy_1 = d^2$$

and

$$xx_1 + (xy_1 + x_1y) \cos \omega + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

90. *Transformation of coordinates.*

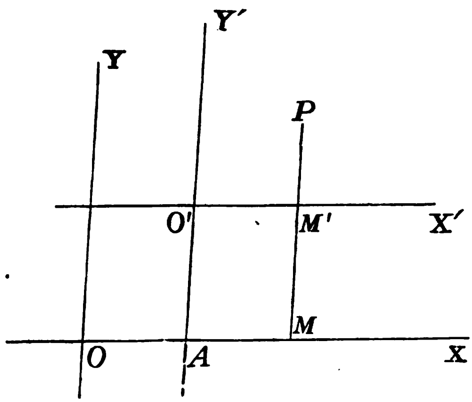
Since the same point may be referred to different systems of coordinates, there must be definite relations connecting these various systems.

We will investigate those relations which are most practically useful, and then shew how to transform the coordinates of a point from any one system to any other.

91. *To change the origin from one point to another, the direction of the axes remaining unaltered.*

Let OX, OY be the old axes, $O'X'$, $O'Y'$ the new: let (x, y), (x', y') be the coordinates of the *same* point P referred to the old and new systems respectively: let h, k be the coordinates of the new origin referred to the old axes, and therefore

$$OA = h, O'A = k, OM = x, PM = y, O'M' = x', PM' = y'.$$



Then $$OM = OA + AM = OA + O'M',$$

$$x = x' + h,$$

$$PM = MM' + PM' = O'A + PM',$$

$$y = y' + k.$$

Ex. Transform the equation $x^2 + y^2 = a^2$, by changing the origin to the point (α, β).

Here $$x = x' + \alpha,$$

$$y = y' + \beta;$$

$$\therefore (x' + \alpha)^2 + (y' + \beta)^2 = a^2,$$

the equation required.

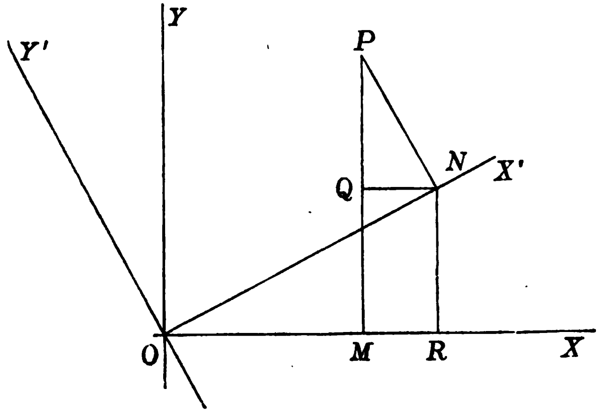
92. *To change the coordinates from one rectangular system to another, the origin being unaltered.*

Let P be the point; x, y, its coordinates referred to

the original axes OX, OY; x', y', referred to the axes OX', OY';

then $$OM = x,\ PM = y,\ ON = x',\ PN = y';$$

let $$XOX' = YOY' = \theta.$$



From N draw NQ, NR perpendicular to PM, OX, respectively; then $NPQ = QNO = NOR = \theta$.

$$\therefore\ OM = OR - RM = OR - NQ = ON \cos\theta - PN \sin\theta,$$

or $$x = x' \cos\theta - y' \sin\theta;$$

so $$PM = MQ + QP = RN + QP = ON \sin\theta + PN \cos\theta,$$

or $$y = x' \sin\theta + y' \cos\theta.$$

Ex. In the equation

$$x^2 - y^2 = a^2,$$

turn the axes through an angle $-45°$.

The equation becomes

$$\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 = a^2,$$

or $$2x'y' = a^2.$$

93. *To transform an equation from one rectangular system to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the original axes; next turn these axes through the required angle.

Thus; if h, k be the coordinates of the new origin referred to the old axes, θ the angle between the original and final axes of x, we shall have

$$x = h + x' \cos\theta - y' \sin\theta,$$
$$y = k + x' \sin\theta + y' \cos\theta.$$

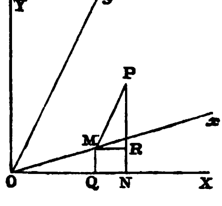
In all these transformations attention must be paid to the *sign* of θ.

The two steps in the transformations should be taken separately.

94. It is obvious that coordinates may be changed from any system to any other: we will now give equations from which any particular transformation may be effected.

We will suppose the two systems to have the same origin, since we have seen that any system may be transferred to parallel axes through the origin (h, k) by simply writing $x' + h$ for x, $y' + k$ for y.

Let Ox, Oy be any axes, P the point x, y, so that $OM = x$, $PM = y$.



Through O draw any straight lines OX, OY at right angles to each other; draw PN perpendicular to OX, and let $ON = X$, $PN = Y$.

Draw MQ, MR perpendicular to ON, PN, respectively: let $xOX = \alpha$, $yOX = \beta$.

Then

$$X = OQ + QN = OQ + MR = x \cos\alpha + y \cos\beta;$$

so $\quad Y = x \sin\alpha + y \sin\beta.$

Now if any other axes be drawn through O, making

angles α', β' with OX, and x', y' be the coordinates of P referred to them, we shall have

$$X = x'\cos\alpha' + y'\cos\beta',\ \ Y = x'\sin\alpha' + y'\sin\beta',$$

$$\therefore\ x\cos\alpha + y\cos\beta = x'\cos\alpha' + y'\cos\beta',$$

$$x\sin\alpha + y\sin\beta = x'\sin\alpha' + y'\sin\beta'.$$

By solving these simultaneous equations, we can get any pair of the quantities x, y, x', y' in terms of the other pair.

Thus if we have to transform from one pair of axes to another with the same origin, we shall always have

$$x = ax' + by',\ y = a'x' + b'y',$$

where a, b, a', b', depend only on the angles the axes make with each other, and not on the position of the point P.

If the origin, as well as the direction of the axes, is changed, these equations will become

$$x = ax' + by' + c,\ y = a'x' + b'y' + c'.$$

Hence we can prove the following important theorem.

95. *The degree of any equation cannot be altered by any transformation of coordinates.*

Let lx^my^n represent the highest term in any equation: let the axes be altered so that

$$x = ax' + by' + c,\ y = a'x' + b'y' + c';$$

then $$lx^my^n = l\,(ax' + by' + c)^m\,(a'x' + b'y' + c')^n.$$

Now there is no term in $(ax' + by' + c)^m$ of a higher degree in x', y', than the m^{th}, or in $(a'x' + b'y' + c')^n$ than the n^{th}: hence there is no term in their product of a higher degree than the $(m+n)^{\text{th}}$.

Hence the degree of an equation cannot be raised by transformation. Neither can it be lowered, for then it could be raised by transforming back again.

The general equation of the first degree $lx + my = d$, represents therefore one distinct class of lines, the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

represents another, and so on.

Examples IX.

The axes are inclined at an angle ω, unless otherwise given.

1. A regular hexagon is described, each side being a: if two adjacent sides be taken as axes, determine equations to the other sides, and the coordinates of the angular points.

2. Two sides of an equilateral triangle are taken as axes: find the equations to the lines which bisect the angles.

3. ABC is a triangle, AB, AC are the axes: find the equations to the straight lines which pass through the angular points, and (i) bisect the opposite sides, (ii) are perpendicular to the opposite sides.

4. Find the equations to the straight lines which pass through a given point (a, b) and of which the portion between the axes is of a fixed length c.

5. Find the straight lines which pass through the point $(a, 0)$, and are perpendicular to the axes.

6. Find the straight lines which pass through $(0, b)$ and make angles α with the axis of x.

7. Find the straight lines which pass through $(0, b)$ and make angles β with the axis of y.

8. The straight lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, will be perpendicular to each other if

$$AA' + BB' = (AB' + A'B) \cos \omega.$$

9. The same straight lines will be equally inclined to the axis of x in opposite directions if

$$\frac{B}{A} + \frac{B'}{A'} = 2 \cos \omega.$$

10. If the straight lines

$$x \cos \alpha + y \cos \beta = p, \quad x \cos \alpha' + y \cos \beta' = p'$$

have equal portions intercepted between the axes

$$p \cos \alpha' \cos \beta' = p' \cos \alpha \cos \beta.$$

11. If $Ax + By + C = 0$, $x \cos \alpha + y \sin \alpha = p$, represent the same straight line then

$$\frac{A}{\cos \alpha} = \frac{B}{\cos \beta} = \frac{-C}{p}$$

and

$$p\{A^2 + B^2 - 2AB \cos(\alpha + \beta)\}^{\frac{1}{2}} + C \sin(\alpha + \beta) = 0.$$

12. If the straight lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1,$$

are perpendicular to each other

$$\cos \omega = \frac{2ab}{a^2 + b^2}.$$

13. The equation

$$x^2 + 2xy \cos \omega + y^2 \cos 2\omega = 0$$

represents two straight lines through the origin which make equal angles with the axis of x, and are perpendicular to each other.

14. If the angle between

$$\frac{x}{l} = \frac{y}{m} \text{ and } \frac{x}{l'} = \frac{y}{m'}, \text{ be } \frac{\pi}{2} - \omega,$$

then

$$(ll' + mm') \cos \omega + (lm' + l'm) \cos^2 \omega \pm (lm' - l'm) \sin^2 \omega = 0.$$

15. The equations $x - y = 0$, $x + y = 0$, represent the bisectors of the angles between the axes, whatever the inclination of the axes may be.

16. Find the equation to the circle when the coordinates of the centre are (h, k) and the radius is r.

17. Find the coordinates of the centre and the radius of the circle whose equation is

$$x^2 + 2xy \cos \omega + y^2 + 2gx + 2fy + c = 0.$$

18. Find the equation to the circle which passes through the origin and the points $(a, 0)$, $(0, b)$: find also the coordinates of the centre and the radius.

19. If the straight line $lx + my = d$ touches

$$x^2 + 2xy \cos \omega + y^2 = a^2,$$

then $$a^2(l^2 + m^2 - 2lm \cos \omega) = d^2 \sin^2 \omega.$$

20. If $Ax + By + C = 0$ touch

$$x^2 + 2xy \cos \omega + y^2 = hx + ky,$$

then

$$A^2k^2 + ABhk + B^2h^2 + 2(Ak + Bh)C \cos \omega = (Ah + Bk)C + C^2 \sin^2 \omega.$$

21. Transform the equations

$$x + y = 2, \quad x + y + 2 = 0, \quad x = y, \quad x + y = 0,$$
$$x^2 + y^2 = 1, \quad x^2 - y^2 = 1, \quad y^2 = 4x,$$

by changing the origin to the point (1, 1).

22. Transform the equations

$$x = y, \quad x + y = a, \quad x^2 + y^2 = a^2, \quad x^2 - y^2 = a^2$$

which are in rectangular coordinates, by turning the axes through an angle $\dfrac{\pi}{4}$.

23. Transform the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$, by changing the origin to the point $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and then turning the axes through an angle θ, such that $\tan \theta = \dfrac{a}{b}$.

24. If in the equation $ax^2 + 2hxy + by^2 = 0$, the rectangular axes be turned through an angle such that the coefficient of x^2 is b, then that of y is a, and that of xy is unaltered.

25. If (xy) $(x'y')$ be the coordinates of the same point referred to two systems YOX, $Y'OX'$ such that $Y'OX' = \omega$, and OX, OY bisect the angles $Y'OX'$ and its supplement, then

$$x = (x' + y') \cos \frac{\omega}{2}, \quad y = (x' - y') \sin \frac{\omega}{2}.$$

Change the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ when

$$Y'OX' = 2 \tan^{-1} \frac{b}{a}.$$

26. If two oblique systems have the same origin and axis of x, then

$$x = x' + y' \frac{\sin(\omega - \omega')}{\sin \omega}, \quad y = y' \frac{\sin \omega'}{\sin \omega},$$

ω, ω' being the angles between the axes.

INDEX TO FORMULAE, &c.

The Point.

The Straight Line.

The Circle.

ANSWERS.

. b. p. 8.

1. $\sqrt{13}$, 13, 41, $5a$, $\sqrt{5b}$, a. 2. $\sqrt{2}$, $7\sqrt{2}$, $2\sqrt{2}$, 2, 2, 5, 13, $\sqrt{h^2+16k^2}$, $(a-b)\sqrt{2}$, $10a$. 3. $\left(\frac{7}{2}, \frac{7}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(0, 0)$, $(0, -1)$, $(0, 1)$, $\left(-1, -\frac{1}{2}\right)$, $\left(5, -\frac{5}{2}\right)$, $\left(\frac{3h}{2}, -k\right)$, $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, $(-3a, -2a)$.

4. Substitute the given coordinates in the expressions $\frac{2x_2+x_1}{3}$, $\frac{2y_2+y_1}{3}$, $2x_2-x_1$, $2y_2-y_1$. 6. (i) 6. (ii) 3. (iii) 6. (iv) 2. (v) 14. (vi) 12. (vii) 12. (viii) $bh-ak$. (ix) $2ab$. (x) $\frac{a^2}{2}\sin(\alpha-\beta)$.

7. $\pm\frac{1}{2}\{x_2y_1-x_1y_2+x_3y_2-x_2y_3\}$.

8. $\pm\frac{1}{2}\{x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_4y_3-x_3y_4+x_1y_4-x_4y_1\}$.

I. d. p. 10.

1. $\theta=\frac{\pi}{4}$, $r=a$, $r^2\cos 2\theta=a^2$, $r\sin^2\theta=4a\cos\theta$, $r\cos(\theta-\alpha)=p$.

2. $x+y=0$, $x^2+y^2=c^2$, $x+y=l$, $2xy=a^2$. $x^2+y^2=2ax$, $x=a$.

I. e. p. 12.

1. Give n values from 0 to 7 in $x=a\cos\frac{n\pi}{4}$, $y=a\sin\frac{n\pi}{4}$.

2. Give n values from 0 to 7 in

$$x=a\sec\frac{\pi}{8}\cos\frac{2n+1}{8}\pi,\quad y=a\sec\frac{\pi}{8}\sin\frac{2n+1}{8}\pi.$$

3. $0, 0$; $a, 0$; $\frac{3}{2}a, \frac{\sqrt{3}}{2}a$; $a, \sqrt{3}a$; $0, \sqrt{3}a$; $-\frac{a}{2}, \frac{\sqrt{3}}{2}a$. 4. R and r being the radii of the circum- and in-circles, the coordinates are (1) $R\sin C, R\cos C$; $r\cot\frac{A}{2}, r$; (2) $R, \frac{\pi}{2}-C$; $r\operatorname{cosec}\frac{A}{2}, \frac{A}{2}$.

5. $\frac{h_2+h_3}{2}, \frac{k_2+k_3}{2}$, &c. 6. $\frac{h_1+h_2+h_3}{3}, \frac{k_1+k_2+k_3}{3}$.

7. $\theta=\tan^{-1}\frac{r_1\sin\theta_1+r_2\sin\theta_2}{r_1\cos\theta_1+r_2\cos\theta_2}$, $r=\frac{1}{2}\{r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)\}^{\frac{1}{2}}$.

8. (1) $\frac{ab}{2}$. (2) $\frac{\sqrt{3}a^2}{4}$. (3) $\frac{5\sqrt{3}}{8}a^2$. (4) $\frac{a^2}{4}(2-\sqrt{3})$. (5) $\frac{\sqrt{3}a^2}{4}$.

II. b. p. 23.

2. The angles are $\frac{\pi}{2}, \frac{\pi}{2}, 0, 0, \frac{\pi}{4}$, $\tan^{-1}\left(-\frac{2}{3}\right)$, $\tan^{-1}\left(-\frac{4}{3}\right)$, $\tan^{-1}\left(\frac{4}{3}\right)$, $\tan^{-1}\left(\frac{4}{3}\right)$, $\tan^{-1}\left(-\frac{4}{3}\right)$, $\tan^{-1}\left(-\frac{3}{2}\right)$, $\tan^{-1}\left(-\frac{2}{3}\right)$, $\tan^{-1}\left(\frac{3}{2}\right)$, $\tan^{-1}\left(-\frac{3}{2}\right)$, $\tan^{-1}\left(\frac{3}{2}\right)$, and the lengths of the perpendiculars are 2, 3, 1, 4, 0, 0, 1 in the next 4 lines, and $\frac{6}{\sqrt{13}}$ in the remainder. 4. θ in each case is $\frac{\pi}{2}$ less than the angle given in the corresponding answer to question 2.

II. c. p. 29.

2. $x+y=a+b$: $kx=hy$: $x=h$: $y+k=0$: $y=x+1$: $7x-5y=5$: $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$. 3. $\sqrt{3}(y-2)=x+1$.

4. $y=x$. 5. $x\cos\alpha-y\sin\alpha=b$. 7. $\sqrt{2}$. 8. $3-2\sqrt{3}$.

9. $\frac{mh+c-k}{\sin\alpha-m\cos\alpha}$. 10. If θ be the angle required it is given by the equation $\sin\theta-m\cos\theta=\frac{mh+c-k}{a}$. 11. $\alpha\pm\beta$.

14. $br\cos\theta+ar\sin\theta=ab$: $r(\cos\theta+\sin\theta)=\frac{\sqrt{3}-1}{2}$: $r\sin\theta=\frac{\sqrt{3}}{2}a$.

15. $\theta=\frac{\pi}{2}-B$, $r\cos(\theta-A)=c\cos A$, $r\cos\theta=b\cos A$,

$$\theta=\frac{\pi}{2}-B,\quad r=a\cot A.$$

III. a. p. 33.

1. (i) $75°$. (ii) $\tan^{-1}\frac{1}{3}$. (iii) $\tan^{-1}\frac{3}{4}$. (iv) $\tan^{-1}\frac{a^2-b^2}{2ab}$. (v) $\tan^{-1}\frac{2(x_1y_2-x_2y_1)}{x_2^2-x_1^2+y_2^2-y_1^2}$. (vi) $2a$. 2. (i) 1, 2, 3. (ii) $\frac{3}{2}$, 2, $\frac{7}{4}$. (iii) $\frac{3}{4}$, $\frac{1}{2}$, -2. (iv) $\frac{1}{3}$, $\frac{1}{2}$, -1. (v) $\frac{1}{3}$, $\frac{1}{3}$, $-\frac{3}{4}$. (vi) The first two are at right angles, the other angles are $\tan^{-1}2+\alpha-\frac{\pi}{2}$ and its complement. 3. $y=2x-4$. 4. $3x+4y=4$. 5. $\frac{x}{3}+\frac{y}{4}+4=0$.
6. $Ax+By=10A$, $Ax+By=Aa+Bb$, $Ax+By=0$.
7. $y=mx\pm a\sqrt{1+m^2}$. 8. $8x+7y=15$. 9. $2x-3y+8=0$.
10. $x+2y=5$, $2x+y=4$. 11. $Bx=A(y-b)$. 12. $ax-by=a^2-b^2$.
13. $ax=by$. 14. $y\cos\alpha-x\sin\alpha=q$. 15. $\sqrt{3}x+y=0$, $x+\sqrt{3}y=0$.
16. $x=0$, $y+\sqrt{3}x=0$. 17. $(a^2-b^2)y=(a\pm b)^2(x-a)$.
18. $(y-b)\tan(\alpha\pm\beta)=x$.

III. b. p. 39.

1. $2\sqrt{2}$. 2. $\frac{3}{\sqrt{13}}$. 3. $\frac{4}{\sqrt{10}}$. 4. $\frac{11}{5}$. 5. $\frac{3ab}{\sqrt{a^2+b^2}}$.
6. $\frac{a^2-b^2}{\sqrt{a^2+b^2}}$. 7. $\frac{Ah+Bk+C-D}{\sqrt{A^2+B^2}}$. 8. $\frac{c^2}{\sqrt{h^2+k^2}}$.
9. $\frac{h^2+k^2-c^2}{\sqrt{h^2+k^2}}$. 10. $a\sqrt{1+m^2}$. 11. $\frac{b^2h^2+a^2k^2-a^2b^2}{(b^4h^2+a^4h^2)^{\frac{1}{2}}}$.
12. $(0, 1)$, $\left(\frac{30}{13}, -\frac{6}{13}\right)$, $\left(\frac{2}{5}, -\frac{1}{5}\right)$, $\left(-\frac{58}{25}, \frac{31}{25}\right)$, $\left(\frac{b^3}{a^2+b^2}, \frac{a^3}{a^2+b^2}\right)$, $\left(\frac{2a^2b}{a^2+b^2}, \frac{2ab^2}{a^2+b^2}\right)$, $\frac{(D-C)A+(Bh-Ak)B}{A^2+B^2}$, $\frac{(D-C)B-(Bh-Ak)A}{A^2+B^2}$. $\left(\frac{hc^2}{h^2+k^2}, \frac{kc^2}{h^2+k^2}\right)$, the same point, $\left(0, \frac{a}{m}\right)$, $\frac{\{a^2(b^2+hk)-b^2h^2\}a^2k}{b^4h^2+a^4k^2}$, $\frac{(b^2(a^2+hk)-a^2k^2)b^2h}{b^4h^2+a^4k^2}$. 13. Treat the equations in pairs as simultaneous. 14. $(a-b)\sin\alpha$, $\frac{4ab}{\sqrt{a^2+b^2}}$, $\frac{7}{\sqrt{41}}$, $\frac{3}{\sqrt{2}}$, $\frac{6}{5}$. 16. c^2.
17. $\frac{a^2}{6}$. 18. $\frac{2a^2+5ab+2b^2}{6}$. 19. $\frac{ma^2+(m^2+1)ab+mb^2}{2(1-m^2)}$.

20. $\frac{7}{30}$. 21. Let $\frac{a^2}{2}$ be the given area, $y-k=m(x-h)$ one of the lines required, then m is one of the roots of the equation

$$(k-mh)^2+ma^2=0.$$

22. $4x-3y=1$, $3x+4y=7$. 23. $\left(\frac{4}{5}, -\frac{2}{5}\right)$. 24. The points are the intersections of the straight lines, $y=mx+c\pm(m^2+1)^{\frac{1}{2}}d$, $x\cos\alpha+y\sin\alpha-p=\pm d$. 25. $\frac{ma+b}{\sqrt{1+m^2}}$, $\frac{a+mb+\sqrt{1+m^2}c}{\sqrt{1+m^2}}$.

26. $\frac{a(b-1)}{a+b}$, $\frac{b(a+1)}{a+b}$, $\frac{7}{a}+\frac{9}{b}=4$. 27. $\frac{4-b}{a+b}a$, $\frac{4+a}{a+b}b$, $a=2$, $b=1$.

28. Bisected. 29. $3y+x=8$, $y-2=(x-2)\tan\left(\frac{\pi}{3}-\alpha\right)$, $y-1=(x-5)\tan\left(\frac{2\pi}{3}-\alpha\right)$, where $\tan\alpha=\frac{1}{3}$. 30. Take the angular points in the order given as A, B, C, D, the equations to the sides AB, BC, CD, DA, are $ay=2b(x-a)$, $bx=2a(y-b)$, $a(y-b)=2bx$, $b(x-a)=2ay$, and to the diagonals AC, BD, $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{a}=\frac{y}{b}$, respectively.

IV. a. p. 46.

1. (i) $x=\frac{a}{2}$, $y=\frac{a}{2}$. (ii) $(1-m)y=(1+m)x$, $(1+m)y+(1-m)x=2c$.
(iii) $x=y$, $7(x+y)=10$. (iv) $bx+ay-ab=\pm(a^2+b^2)^{\frac{1}{2}}(x-a)$.
(v) $y=x\tan\frac{\alpha+\beta}{2}$, $x(\cos\alpha+\cos\beta)+y(\sin\alpha+\sin\beta)=2p$.
(vi) $\sqrt{5}(x+y-2)=\pm(x+3y-4)$.
(vii) $(a-b)x=(a+b)y$, $(a+b)x+(a-b)y=2ab$.
(viii) $\frac{x+1-y}{\sqrt{2}}=\pm\frac{2(x+1)-y}{\sqrt{5}}$. 2. $11x=32y$.

3. $(ac'-a'c)x+(bc'-b'c)y=0$. 4. $x=y$. 5. $x+y=3$.
6. $y=k$. 7. $(ma+b)y=mab$. 8. $19x=35$. 9. $7x-y=18$.
10. $(l_1m_2-l_2m_1)(l_3x+m_3y)=d_1(l_2m_3-l_3m_2)+d_2(l_1m_3-l_3m_1)$.
11. $2x+3y+19=0$. 12. $13(3x+y)=58$. 13. $x+4y=16$.
14. $(ma+b)(ax-by)=(a-mb)ab$. 15. $16x+3y=35$.
16. $x(\cos\alpha+\lambda\cos\beta)+y(\sin\alpha+\lambda\sin\beta)=p(1+\lambda)$ where

$$(1+\lambda)p(1-m)=c\{\cos\alpha+\sin\alpha+\lambda(\cos\beta+\sin\beta)\}.$$

17. (i) the origin. (ii) 1, 1. (iii) 2, 2. (iv) $\frac{a}{2}$, $\frac{b}{2}$. (v) 0, $-c$, (vi) 2, -1. 19. $5x+9y=17$, $5x+y=13$. 20. $7x=9y$, $x=7y$.

21. $$(x\cos\alpha+y\sin\alpha)(a-b)=(p_1-p_2)x+ap_2\quad bp_1$$
$$(x\cos\alpha+y\sin\alpha)(a-b)=(p_2-p_1)x+ap_1-bp_2.$$

22. $a-h,\ b-k,\ \dfrac{x}{a}+\dfrac{y}{b}=1,\ (a-2h)y-(b-2k)x=ak-bh.$

IV. b. p. 54.

1. (i) $\dfrac{a}{3},\ \dfrac{b}{3}$. (ii) $0, 0$. (iii) $\dfrac{a}{3},\ 0$. 2. (i) $1, 1$. (ii) $0, 2$. (iii) $1,\ \dfrac{1}{3}$. (iv) $\dfrac{a}{3}\left\{1+\dfrac{b}{ma+b}+\dfrac{1}{m^2+1}\right\},\ \dfrac{ma}{3}\left\{\dfrac{b}{ma+b}+\dfrac{1}{m^2+1}\right\}$.

4. Orthocentre $h,\ \dfrac{a^2-h^2}{k}$, centroid $\dfrac{h}{3},\dfrac{k}{3}$. 5. Orthocentres $0, 0$; $0, 0$; $1, 1$; $\dfrac{a}{1+m^2},\ \dfrac{ma}{1+m^2}$; circumcentres $\dfrac{3}{2},\ \dfrac{3}{2}$; $0, 3$; $1, 0$; the point of intersection of $2y=m(2x-a)$ and $2(ma+b)(ax+by)=ma^2+2ab-mb^2$.

6. $\dfrac{mb^2}{ma+b},\ \dfrac{mab}{ma+b}$; $\dfrac{4}{9},\ \dfrac{4}{9}$; $a-mb,\ b$; $\{p\cos\alpha+b(\sin\alpha+\cos^2\alpha)\}$, 0.

7. $0,\ \dfrac{2}{3}$; $\dfrac{12}{13},\ \dfrac{57}{26}$; the incentre is determined by the equations
$$\frac{4y+3x+5}{5}=\frac{11+2x-5y}{\sqrt{29}}=\frac{7-5x+y}{\sqrt{26}}.$$

8. (i) $x+y=\sqrt{3}-1,\ x(\sqrt{3}+1)-y(\sqrt{3}-1)=4,\ y=0,\ 2(\sqrt{3}-1),\ 0$. (ii) $x=y,\ \sqrt{2}x=2-x-y,\ \sqrt{2}y=2-x-y,\ 2-\sqrt{2},\ 2-\sqrt{2}$. (iii) $x+y=2$, $3y-x=2,\ x=1,\ 1,\ 1$. (iv) $y=1,\ x+y=\dfrac{3}{4},\ y-x=\dfrac{3}{4},\ \dfrac{1}{4},\ 1$. (v) $y=\dfrac{3}{2}$, $7x+y=24,\ x-y=\dfrac{12}{7},\ \dfrac{45}{14},\ \dfrac{3}{2}$. (vi) $x(1+\cos\alpha)+y\sin\alpha=0$, $x(1+\cos\alpha)-y\sin\alpha=0,\ y=0$; $0, 0$. 9. $\dfrac{x+y-3}{\sqrt{2}}=y,\ \sqrt{5}y=y-2x$, $\dfrac{x+y-3}{\sqrt{2}}+\dfrac{y-2x}{\sqrt{5}}=0$.

IV. c. p. 59.

1. $x+y=1,\ 2y=x+1,\ 3y+2x=4$. 5. (i) the axes. (ii) the lines $x=\pm y$. (iii) the axes, and $x=y$. (iv) $Ax+By+1=0,\ Bx+Ay=0$. (v) $x\cos\theta\pm y\sin\theta=\pm p$. (vi) $x+2y=0,\ x=\pm\sqrt{2}y$. (vii) $x+1=3y$, or $x+y+1=0$. (viii) The point $(6, 3)$. (ix) $x-y=3$, or $x-3y+3=0$. (x) $2x+y-3=0$, or $x-2y+3=0$. 6. The 4 points for which $r=\pm a,\ \theta=\pm\alpha$; the point (r, α); the points (a, β), (b, α).

11. (i) $(ab'-a'b)^2=4(ah'-a'h)(hb'-h'b)$;
(ii) $(aa'-bb')^2+4(bh'+a'h)(hb'+h'a)=0$; (iii) $\dfrac{h^2-ab}{(a+b)^2}=\dfrac{h'^2-a'b'}{(a'+b')^2}$.

12. $a=b$.

V. p. 67.

1. $3a, 2a, (a^2+b^2)^{\frac{1}{2}}$. **2.** $a, -a, (2a^2+c^2)^{\frac{1}{2}}; -\frac{a}{2}, -\frac{b}{2}, \frac{\sqrt{5(a^2+b^2)}}{2}; \frac{3}{2}, 2, \frac{3}{2}; -4, -3, 5; 3, 4, 6; \frac{a}{2}, 0, \frac{a}{2}; 0, \frac{b}{2}, \frac{b}{2}; \frac{a}{2}, 0, \frac{\sqrt{5}a}{2}; \frac{a}{2}, \frac{b}{2}, \frac{(a^2+b^2)^{\frac{1}{2}}}{2}$. **3.** $x^2+y^2-2ax+2ay+a^2=0$.
4. $x^2+y^2=6ax+8ay$. **5.** $x^2+y^2+2b^2+c^2=2b(x+y)+2c(x-y)$.
6. $x^2+y^2=ax+by$. **7.** $7(x^2+y^2)=25(x+y)$; $x^2+y^2=7y-x$; $x^2+y^2=5y$; $h(x^2+y^2)=(h^2+k^2)x$. **8.** $x^2+y^2-11x+9=0$; $2(x^2+y^2)-11y+7=0$; $x^2+y^2-2x+3y=0$; $9(x^2+y^2)+21x-y-78=0$.
9. $x^2+y^2+6=4(x+y)$. **10.** The coordinates of the centres are $\frac{36}{5}, \frac{31}{5}$ or $-\frac{36}{5}, -\frac{1}{5}$. **11.** $x^2+y^2=\pm 2a(x\pm y)$. **12.** $x^2+y^2=\sqrt{2}ay$.
13. $\mu(x^2+y^2-ax+by)=ay-bx+ab$. **14.** $g=28, 5\sqrt{37}$. **15.** $\sqrt{5}$.
16. $x^2+y^2=2(x+y)+8$. **17.** $(a^2+b^2)(x^2+y^2-c^2)=a^2b^2$.
19. $x^2+y^2=2by+a^2$. **20.** $x^2+y^2=2$.
23. $x^2+y^2-\frac{a\sqrt{3}}{3}y=\frac{a^2}{4}$; $r=2\sqrt{3}a\cos\left(\theta-\frac{\pi}{6}\right)$.

VI. a. p. 75.

1. $x=d$, $3x-4y=5d$, $ky-hx=d^2$. **2.** (i) $x\sin\alpha-y\cos\alpha=\pm d$, where α is the given angle; (ii) $kx+hy=\pm d(h^2+k^2)^{\frac{1}{2}}$; (iii) $Bx-Ay=\pm d(A^2+B^2)^{\frac{1}{2}}$; (iv) $d(y-b)=\pm\sqrt{b^2-d^2}x$, where (O, b) is the given point; (v) $x\cos\theta+y\sin\theta=d$, where $c\cos(\alpha-\theta)=d\pm\delta$; (vi) $x\cos\theta+y\sin\theta=d$, where $2d^2=a^2\sin 2\theta$. **3.** $C^2=(A^2+B^2)d^2$, $(x_1\sin\alpha-y_1\cos\alpha)^2=d^2$, $c^2=(1+m^2)d^2$. **4.** $\pm\sqrt{3}, \mp\sqrt{3}, 1$.
5. $d^2(l^2+m^2)=1$. **7.** $l=\pm 4$. **8.** $m=\pm\sqrt{3}$. **9.** 1.
10. 4, 3 and 3, 4. **11.** $4x+3y=25$, $3x+4y=25$.
13. $\frac{(x_2-x_1)d^2}{x_2y_1-x_1y_2}, \frac{(y_2-y_1)d^2}{x_2y_1-x_1y_2}$.

VI. b. p. 82.

1. $3x-4y=0$ and $3x-4y=25$; $2x-y=0$ and $2x-y+10=0$; $x=0$ and $x=a$; $y=0$ and $y=b$; $ax+by=0$ and $ax+by+a^2+b^2=0$.
2. $-a, -b$; $2a, b$; $0, c$; $c, 0$; $c(1+\cos\theta), c(1+\sin\theta)$; $0, b$.

3. $(a\sin\theta - b\cos\theta)^2 + 2p(a\cos\theta + b\sin\theta) = p^2$. 4. $y = \pm\frac{a}{2}$, $x=0$, $x=a$; $y=0$, $y+2b=0$, $x=\pm b$; $y=0$ or $2a$, $x=0$ or $2a$; $y=0$ or $2b$, $x=a\pm b$; $y=k\pm a$, $x=h\pm a$; $y=0$ or -6, $x=5$ or -1; $y=4$ or -2, $x=4$ or -2. 5. The values of x range from -2 to 8, and the corresponding values of y are easily obtained by putting the equation in the form $(x-3)^2+(y-4)^2=25$; then write the values for x' and y' in the equation $xx'+yy'=3(x+x')+4(y+y')$. 6. 5 or -3.

7. $a=\frac{77}{12}$ or -4, $b=4$ or $\frac{288}{7}$, $c=27$ or -23. 8. $x^2+y^2=\pm\sqrt{2}ax$ and $x^2+y^2=\pm\sqrt{2}ay$. 9. (i) $c=3$. (ii) $c=0$ or 4.

10. $c(x^2+y^2)=(8x+6y)\left(\frac{x}{4}+\frac{y}{3}\right)$ or $6c(x^2+y^2)=12x^2+25xy+12y^2$, which has equal roots if $4(6c-12)^2=25^2$; $c=\frac{49}{12}$ or $-\frac{1}{12}$.

11. $x^2+y^2-2m(ax+by)\left(\frac{x}{b}+\frac{y}{a}\right)+c\left(\frac{x}{b}+\frac{y}{a}\right)^2=0$, which has equal roots if $8m^2a^2b^2-2mab(a^2+b^2+c)+(a^2+c)(b^2+c)=0$. 12. 164.

13. $x^2+y^2+\frac{2a}{1+\cot a}(x-y)+\frac{a^2}{(1+\cot a)^2}=0$. 14. $bg=fa$.

16. $(p+2g\cos a)(p+2f\sin a)=(g\sin a+f\cos a)^2$.

19. $(A^2+B^2)((x-h)^2+(y-k)^2)=(Ah+Bk+C)^2$. 20. $x^2+y^2-5x+9=0$.

21. $b(x^2+y^2)=(a^2+b^2)y$. 24. $\theta-a=\frac{\pi}{2}$, $r\cos(\theta-a)=l$.

25. The values of θ are $\frac{\pi}{4}+\frac{a}{2}+\frac{\beta}{2}$, $\frac{a}{2}+\frac{\beta}{2}-\frac{\pi}{4}$.

VII. a. p. 90.

1. $x+y=d$, $3x-2y=d$, $ax=d^2$, $by-d^2=0$. 2. $\frac{d^2}{a}$, 0; 0, $\frac{d^2}{b}$; $-\frac{d^2A}{C}$, $-\frac{d^2B}{C}$; $\frac{d^2}{a}$, $\frac{d^2}{b}$. 11. $2x+8y=25$, $8x+2y=25$.

15. $2gx+2fy+g^2+f^2=0$, $2gx+2fy+g^2+f^2+c=0$.

16. $-\frac{ag^2+bfg-abf}{af+bg+ab}$, $-\frac{bf^2+afg-abf}{af+bg+ab}$.

VII. b. p. 96.

3. (i) a straight line. (ii) circle. (iii) a circle on a diameter through O perpendicular to the locus of P. (iv) a straight line perpendicular to the diameter through O. (v) a circle. 4. (i) $x=\frac{a}{4}$, (ii) $3(x^2+y^2+a^2)=10ax$, (iii) $x^2+y^2=\frac{c^2}{2}$, (iv) $(1+m)x=(1-m)a$.

5. A straight line. **6.** A straight line. **7.** Straight lines. **9.** A circle. **11.** The circle $(x-h)^2+(y-k)^2+b(x\cos a+y\sin a)=bp$. **12.** A straight line parallel to the loci of B and C. **13.** A straight line. **14.** A straight line.

VIII. p. 105.

1. (i) $2x-8y+16=0$: (ii) $ax=by$: (iii) $2y+35=0$. **2.** (i) 8, 4: (ii) $-a$, $-b$: (iii) 24, 10. **4.** $x+y=3$, $4x-2y=1$, $2x-4y+5=0$, $\frac{7}{6}, \frac{11}{6}$. **6.** $x^2+y^2-4=m(x^2+y^2+8-6(x+y))$, $x^2+y^2=2(x+y)$.

IX. p. 118.

1. $y=0$, $x-y=a$, $x=2a$, $y=2a$, $y-x=a$, $x=0$. $0, 0$; $a, 0$; $2a, 0$; $2a, 2a$; $a, 2a$; $0, a$. **2.** $x=y$, $x+2y=a$, $2x+y=a$, where a is the length of a side. **3.** With the usual notation in Trigonometry, the medians are $\frac{x}{c}=\frac{y}{b}$, $\frac{2x}{c}+\frac{y}{b}=1$, $\frac{x}{c}+\frac{2y}{b}=1$, and the perpendiculars

$$x+y\cos A=b\cos A,\ x\cos A+y=c\cos A,\ x\cos B=y\cos C.$$

4. $y-b+m(x-a)=0$, where $(b+ma)^2=m^2c^2/(1-2m\cos\omega+m^2)$.

5. $x+y\cos A=a$, $x+y\sec A=a$. **6.** $y-b=\mp\frac{x\sin a}{\sin(\omega\pm a)}$.

7. $(y-b)\sin\beta\pm x\sin(\omega\pm\beta)=0$.

16. $(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=r^2$.

17. The coordinates of the centre are

$$-(g-f\cos\omega)\operatorname{cosec}^2\omega,\quad -(f-g\cos\omega)\operatorname{cosec}^2\omega$$

and the radius is

$$\{(g^2+f^2-2gf\cos\omega)\operatorname{cosec}^2\omega-c\}^{\frac{1}{2}}.$$

18. $x^2+y^2+2xy\cos\omega=ax+by$ the coordinates and radius are obtained as in the preceding question.

21. $x'+y'=0$, $x'+y'+4=0$, $x'=y'$, $x'+y'+2=0$, $x'^2+y'^2+2x'+2y'+1=0$, $x'^2-y'^2+2(x'-y')=1$, $y'^2+2y'=4x'+3$. **22.** $y'=0$, $x'=\frac{a}{\sqrt{2}}$, $x'^2+y'^2=a^2$, $2x'y'+a^2=0$. **23.** $y'=0$. **25.** $4x'y'=a^2+b^2$.

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